

# Random evolutionary games and random polynomials

Manh Hong Duong<sup>1</sup> and The Anh Han<sup>2</sup>

<sup>1</sup>School of Mathematics, University of Birmingham, UK

<sup>2</sup>School of Computing, Engineering and Digital Technologies, Teesside University, UK

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## Abstract

In this paper, we discover that the class of random polynomials arising from the equilibrium analysis of random asymmetric evolutionary games is *exactly* the Kostlan-Shub-Smale system of random polynomials, revealing an intriguing connection between evolutionary game theory and the theory of random polynomials. Through this connection, we analytically characterize the statistics of the number of internal equilibria of random asymmetric evolutionary games, namely its mean value, probability distribution, central limit theorem and universality phenomena. Moreover, comparing symmetric and asymmetric random games, we establish that symmetry in the group interaction increases the expected number of internal equilibria. Our research establishes new theoretical understanding of asymmetric evolutionary games and why symmetry/asymmetry of group interactions matters.

## 1 Introduction

Statistics of roots of (systems of) random polynomials has become a topic of active research over the last century, dating back at least to seminal papers [BP32, LO39, Kac43, LO45, LO48]. The topic provides an everlasting source of challenging mathematical problems, leading to the developments of powerful methods and techniques in analysis, combinatorics and probability theory; see recent papers [TV15, DNV15, DNV18, NNV16, NV21, NV22] and references therein for the latest results of the field. It has also found applications in the study of complex phenomena/systems in many other disciplines, such as quantum chaotic dynamics [BBL92] and quantized vortices in the ideal Bose gas [CHS<sup>+</sup>06]) in physics, the theory of computational complexity [SS93], feasibility and stability of ecological systems [May73, AS20], persistence and first-passage properties in non-equilibrium systems [SM07, SM08, BMS13], steady states of chemical reaction networks [FS22] and the gradients of deep linear networks [MCTH21]. An important class of random polynomials studied intensively in the literature is the Kostlan-Shub-Smale (also known as elliptic or binomial) random polynomials in which the variance of the random coefficients are binomials, cf. Section 2.3. According to [EK95] “This particular random polynomial is probably the more natural definition of a random polynomial”. In the present work, we show that it arises naturally yet from *evolutionary game theory*.

Evolutionary game theory (EGT), which incorporates game theory into Darwin’s evolution theory, constitutes a powerful mathematical framework for the study of dynamics of frequencies of competing strategies in large populations. Introduced in 1973 by Maynard Smith and Price in 1973 [SP73], over the last 50 years, the theory has found its applications in diverse disciplines including biology, physics, economics, computer sciences and mathematics, see e.g. [MS81, NM92, HS<sup>+</sup>98b, NM92, SF07, SP11] and the recent survey [TG23] for more information. Incorporating stochasticity/randomness into evolutionary games is of vital importance to capture

the inevitable uncertainty, which is an inherent property of complex systems due to environmental and demographic noise or may arise from different sources such as lack of data for measuring the payoffs or unavoidable human estimate errors [May73, AT15, CKR21, BAB+23]. A key to gain insightful understanding of the feasibility, diversity and stability of random evolutionary processes is to characterize the statistics of the number of equilibrium points of the dynamics [GT10, HST+18, GF13, SMWN19, HHT15].

Herein we show that the class of random polynomials arising from the study of equilibria of random *asymmetric* evolutionary games, in which the payoff of a player in a group depends on the ordering of its members, is *exactly* the celebrated Kostlan-Shub-Smale system of random polynomials. This is intriguing since in previous works, the study of equilibria of *symmetric* random evolutionary games, in which the payoff of a player in a group interaction is independent of the ordering of its members, gives rise to a different class of random polynomials [GT10, HTG12, DH16, DTH18, DTH19, CDP22, CDP19]. Using this connection, we characterize the statistics (expected number, probability distribution, central limit theorem and universality) of the number of internal equilibria of random asymmetric evolutionary games. We show that symmetry enhances the expected number of internal equilibria. We also numerically investigate universality properties for the number of internal equilibria of symmetric random evolutionary games.

**Organization.** In Section 2 we recall the replicator dynamics for multi-player multi-strategy game and the Kostlan-Shub-Smale system of random polynomials deriving the aforementioned connection. In Section 3 we characterize the statistics of the number of internal equilibria for random asymmetric evolutionary games. In Section 4 we compare symmetric and asymmetric games. Finally we provide further discussions for future work in 5.

## 2 Multi-player multi-strategy games and random polynomials

### 2.1 The replicator dynamics

The classical approach to evolutionary games is replicator dynamics [TJ78, Zee80, HS98a, SS83, Now06], capturing Darwin’s principle of natural selection that whenever a strategy has a fitness larger than the average fitness of the population, it is expected to spread. In the present work, we consider *asymmetric games* where the order of the participants is relevant. As discussed in [MH15] “Biological interactions, even between members of the same species, are almost always asymmetric due to differences in size, access to resources, or past interactions.” Asymmetry also plays a crucial role in social, economic and multi-agent interactions due to the difference in roles and locations of the parties involved, see e.g. [SZ92, Fri98, MH15, TPL+18, HSM19, SAP22, MW22, OEH22]. Models using asymmetric games, instead of symmetric ones, are thus more realistic and representative of real-world interactions.

To describe the mathematical model, we consider an infinitely large population with  $n$  strategies whose frequencies are denoted by  $x_i$ ,  $1 \leq i \leq n$ . The frequencies are non-negative real numbers summing up to 1. The interaction of the individuals in the population takes place in randomly selected groups of  $d$  participants, that is, they play and obtain their fitness from  $d$ -player games. The fitness of a player is calculated as average of the payoffs that they achieve from the interactions using a theoretic game approach. Let  $i_0$ ,  $1 \leq i_0 \leq n$ , be the strategy of the focal player. Let  $\alpha_{i_1, \dots, i_{d-1}}^{i_0}$  be the payoff that the focal player obtains when it interacts with the group  $(i_1, \dots, i_{d-1})$  of  $d-1$  other players where  $i_k$  (with  $1 \leq i_k \leq n$  and  $1 \leq k \leq d-1$ ) be the strategy of the player in position  $k$ . Then the average payoff or fitness of the focal player is given by

$$\pi_{i_0} = \sum_{1 \leq i_1, \dots, i_{d-1} \leq n} \alpha_{i_1, \dots, i_{d-1}}^{i_0} x_{i_1} \dots x_{i_{d-1}}. \quad (1)$$

Given a set of non-negative integer numbers  $\{k_i\}_{i=1}^n$  satisfying  $\sum_{i=1}^n k_i = d - 1$ , let us define

$$\mathcal{A}_{k_1, \dots, k_n} := \left\{ \{i_1, \dots, i_{d-1}\} : 1 \leq i_1, \dots, i_{d-1} \leq n \right. \\ \left. \text{and there are } k_i \text{ players using strategy } i \text{ in } \{i_1, \dots, i_{d-1}\} \right\}.$$

By the multinomial theorem, it follows that

$$|\mathcal{A}_{k_1, \dots, k_n}| = \binom{d-1}{k_1, \dots, k_n} = \frac{(d-1)!}{k_1! \dots k_n!}$$

By re-arranging appropriate terms, Equation (1) can be re-written as

$$\pi_{i_0} = \sum_{\substack{0 \leq k_1, \dots, k_n \leq d-1 \\ \sum_{i=1}^n k_i = d-1}} a_{k_1, \dots, k_n}^{i_0} \prod_{k=1}^n x_k^{k_i} \quad \text{for } i_0 = 1, \dots, n, \quad (2)$$

where

$$a_{k_1, \dots, k_n}^{i_0} := \sum_{\{i_1, \dots, i_{d-1}\} \in \mathcal{A}_{k_1, \dots, k_n}} \alpha_{i_1, \dots, i_{d-1}}^{i_0}. \quad (3)$$

Now the replicator equations for  $d$ -player  $n$ -strategy games can be written as a system of  $n - 1$  differential equations [HS98a, Sig10]

$$\dot{x}_i = x_i (\pi_i - \langle \pi \rangle) \quad \text{for } i = 1, \dots, n - 1, \quad (4)$$

where  $\langle \pi \rangle = \sum_{k=1}^n x_k \pi_k$  is the average payoff of the population. Note that, in addition to the  $n - 1$  equations above,  $\sum_{i=1}^n x_i = 1$  must also be satisfied.

## 2.2 Equilibria of the replicator dynamics

It follows from (4) that the vertices of the unit cube in  $\mathbf{R}^n$  are equilibria of the replicator dynamics. In the following analysis, we focus on *internal equilibria*, which are given by the points  $(x_1, \dots, x_n)$  where  $0 < x_i < 1$  for all  $1 \leq i \leq n - 1$  that satisfy

$$\pi_i = \langle \pi \rangle \quad \text{for all } i = 1, \dots, n.$$

The system above is equivalent to  $\pi_i - \pi_n = 0$  for all  $i = 1, \dots, n - 1$ . Using (2) we obtain a system of  $n - 1$  equations of multivariate polynomials of degree  $d - 1$

$$\sum_{\substack{0 \leq k_1, \dots, k_n \leq d-1, \\ \sum_{i=1}^n k_i = d-1}} b_{k_1, \dots, k_n}^i \prod_{i=1}^n x_i^{k_i} = 0 \quad \text{for } i = 1, \dots, n - 1, \quad (5)$$

where

$$b_{k_1, \dots, k_n}^i := a_{k_1, \dots, k_n}^i - a_{k_1, \dots, k_n}^n \\ = \sum_{\{i_1, \dots, i_{d-1}\} \in \mathcal{A}_{k_1, \dots, k_n}} \left( \alpha_{i_1, \dots, i_{d-1}}^i - \alpha_{i_1, \dots, i_{d-1}}^n \right) \\ = \sum_{\{i_1, \dots, i_{d-1}\} \in \mathcal{A}_{k_1, \dots, k_n}} \beta_{i_1, \dots, i_{d-1}}^i, \quad (6)$$

where  $\beta_{i_1, \dots, i_{d-1}}^i$  denotes the difference of the payoff entries

$$\beta_{i_1, \dots, i_{d-1}}^i := \alpha_{i_1, \dots, i_{d-1}}^i - \alpha_{i_1, \dots, i_{d-1}}^n. \quad (7)$$

Using the transformation  $y_i = \frac{x_i}{x_n}$  (recalling that  $0 < x_n < 1$ ), with  $0 < y_i < +\infty$  and  $1 \leq i \leq n-1$  and dividing the left hand side of (5) by  $x_n^{d-1}$  we obtain the following system of polynomial equations in terms of  $(y_1, \dots, y_{n-1})$

$$\sum_{\substack{0 \leq k_1, \dots, k_{n-1} \leq d-1, \\ \sum_{i=1}^{n-1} k_i \leq d-1}} b_{k_1, \dots, k_n}^i \prod_{i=1}^{n-1} y_i^{k_i} = 0 \quad \text{for } i = 1, \dots, n-1. \quad (8)$$

Noting that  $\{x_i\}_{i=1}^n$  can be computed from  $\{y_i\}_{i=1, \dots, n-1}$  via the transformation

$$x_i = \frac{y_i}{1+y}, \quad i = 1, \dots, n-1 \quad \text{and} \quad x_n = \frac{1}{1+y} \quad \text{where} \quad y = \sum_{i=1}^{n-1} y_i. \quad (9)$$

Thus finding an internal equilibrium of a  $d$ -player  $n$ -strategy evolutionary game using the replicator dynamics is equivalent to finding a positive root of the system of polynomial equations (8). It is noteworthy that (9) is precisely the transformation to obtain the Lotka–Volterra equation for  $n-1$  species from the replicator dynamics for  $n$  strategies [HS98a, PN02].

### 2.3 Kostlan-Shub-Smale system of random polynomials

Kostlan-Shub-Smale [SS93, Kos93, EK95] random polynomials  $\mathcal{P}_{d,m} = (P_1, \dots, P_m)$  consist of  $m$  random polynomials in  $m$  variables with common degree  $\mathbf{d}$

$$P_\ell(\mathbf{x}) = \sum_{|\mathbf{j}| \leq \mathbf{d}} a_{\mathbf{j}}^{(\ell)} \mathbf{x}^{\mathbf{j}}$$

where

- (i)  $\mathbf{j} = (j_1, \dots, j_m) \in \mathbb{N}^m$  and  $|\mathbf{j}| = \sum_{k=1}^m j_k$ ,
- (ii)  $\mathbf{x} = (x_1, \dots, x_m)$  and  $\mathbf{x}^{\mathbf{j}} = \prod_{k=1}^m x_k^{j_k}$ ,
- (iii)  $a_{\mathbf{j}}^{(\ell)} = a_{j_1 \dots j_m}^{(\ell)} \in \mathbf{R}$ ,  $\ell = 1, \dots, m$ ,  $|\mathbf{j}| \leq \mathbf{d}$  are centred random variables,
- (iv)  $\text{Var}(a_{\mathbf{j}}^{(\ell)}) = \binom{\mathbf{d}}{\mathbf{j}} = \frac{\mathbf{d}!}{j_1! \dots j_m! (\mathbf{d} - |\mathbf{j}|)!}$ .

In the univariate ( $m = 1$ ) case, this class of random polynomials is also known as elliptic or normal random variables.

### 2.4 From random evolutionary games to random polynomials

As discussed in the introduction, to obtain more realistic models capturing the unavoidable uncertainty, we consider here *random evolutionary games* where the payoffs entries  $\alpha_{i_1, \dots, i_{d-1}}^i$  (thus all the coefficients  $\beta_{i_1, \dots, i_{d-1}}^i$ ) are random variables. Suppose that  $\{\beta_{i_1, \dots, i_{d-1}}^i, \{i_1, \dots, i_{d-1}\} \in \mathcal{A}_{k_1, \dots, k_n}\}$  are iid centred random variables with unit variance, then it follows from (6) that (8) becomes a system of random polynomial equations whose coefficients are independent centred random variables with variances

$$\text{Var}(b_{k_1, \dots, k_n}^i) = \binom{d-1}{k_1, \dots, k_n}. \quad (10)$$

In particular, if  $\{\beta_{i_1, \dots, i_{d-1}}^i, \{i_1, \dots, i_{d-1}\} \in \mathcal{A}_{k_1, \dots, k_n}\}$  are iid standard Gaussian random variables then  $\{b_{k_1, \dots, k_n}^i\}$  are centred Gaussian random variables with variances given by (10).

It follows that the polynomial system determining internal equilibria in multi-player multi-strategy random asymmetric evolutionary games is *precisely* the Kostlan-Shub-Smale polynomial system. As a consequence, the number of internal equilibria in  $d$ -player  $n$ -strategy asymmetric games is equal to the number of positive roots of the Kostlan-Shub-Smale polynomial system  $\mathcal{P}_{d-1, n-1}$ .

**Lemma 2.1.** *Let  $\mathcal{N}_{d,n}$  be the number of internal equilibria of  $d$ -player  $n$ -strategy asymmetric evolutionary games and  $\mathcal{N}_{d,m}$  be the number of real roots of the Kostlan-Shub-Smale polynomial system. Then*

$$\mathcal{N}_{d,n} = \frac{1}{2^{n-1}} \mathcal{N}_{d-1, n-1}. \quad (11)$$

It is this exact correspondence being the novelty of the present work. This connection paves the way for characterizing the statistics of the number of internal equilibria in multi-player multi-strategy random asymmetric evolutionary games by employing existing techniques and results from the well-established field of random polynomials.

## 2.5 Multi-player two-strategy evolutionary games

In this section, we focus on  $d$ -player two-strategy evolutionary games. In this case, (8) becomes a polynomial equation of degree  $d-1$

$$P_d(y) := \sum_{k=0}^{d-1} b_k y^k = 0, \quad (12)$$

where  $y = \frac{x}{1-x}$  being the ratio of the frequencies of the two strategies and for  $0 \leq k \leq d-1$

$$b_k = \sum_{\{i_1, \dots, i_{d-1}\} \in \mathcal{A}_k} \beta_{i_1, \dots, i_{d-1}} = \sum_{\{i_1, \dots, i_{d-1}\} \in \mathcal{A}_k} \left( \alpha_{i_1, \dots, i_{d-1}}^1 - \alpha_{i_1, \dots, i_{d-1}}^2 \right), \quad (13)$$

where the sums are taken over all  $\binom{d-1}{k}$  sets of  $\{i_1, \dots, i_{d-1}\} \in \mathcal{A}_k$  with

$$\mathcal{A}_k := \left\{ \{i_1, \dots, i_{d-1}\} : 1 \leq i_1, \dots, i_{d-1} \in \{1, 2\} \right. \\ \left. \text{and there are } 0 \leq k \leq d-1 \text{ players using strategy 1 in } \{i_1, \dots, i_{d-1}\} \right\}. \quad (14)$$

**Example 2.1.** We provide concrete examples of asymmetric games to demonstrate the abstract theory.

1. Three-player two-strategy asymmetric game ( $d=3, n=2$ ), with the following payoff matrix

Strategy \ Opposing	2, 2	1, 2	2, 1	1, 1
1	$\alpha_{2,2}^1$	$\alpha_{1,2}^1$	$\alpha_{2,1}^1$	$\alpha_{1,1}^1$
2	$\alpha_{2,2}^2$	$\alpha_{1,2}^2$	$\alpha_{2,1}^2$	$\alpha_{1,1}^2$

Equation (12) can be rewritten as

$$\beta_{2,2} + (\beta_{1,2} + \beta_{2,1})y + \beta_{1,1}y^2 = 0,$$

where

$$\beta_{22} = \alpha_{2,2}^1 - \alpha_{2,2}^2, \quad \beta_{1,2} = \alpha_{1,2}^1 - \alpha_{1,2}^2, \quad \beta_{2,1} = \alpha_{2,1}^1 - \alpha_{2,1}^2, \quad \beta_{1,1} = \alpha_{1,1}^1 - \alpha_{1,1}^2.$$

2. Four-player two-strategy asymmetric game ( $d = 4, n = 2$ ), with the following payoff matrix

Strategy \ Opposing	2, 2, 2	1, 2, 2	2, 1, 2	2, 2, 1	1, 1, 2	1, 2, 1	2, 1, 1	1, 1, 1
	1	$\alpha_{2,2,2}^1$	$\alpha_{1,2,2}^1$	$\alpha_{2,1,2}^1$	$\alpha_{2,2,1}^1$	$\alpha_{1,1,2}^1$	$\alpha_{1,2,1}^1$	$\alpha_{2,1,1}^1$
2	$\alpha_{2,2,2}^2$	$\alpha_{1,2,2}^2$	$\alpha_{2,1,2}^2$	$\alpha_{2,2,1}^2$	$\alpha_{1,1,2}^2$	$\alpha_{1,2,1}^2$	$\alpha_{2,1,1}^2$	$\alpha_{1,1,1}^2$

Equation (12) can be rewritten as

$$P_4(y) = \beta_{2,2,2} + (\beta_{1,2,2} + \beta_{2,1,2} + \beta_{2,2,1})y + (\beta_{1,1,2} + \beta_{1,2,1} + \beta_{2,1,1})y^2 + \beta_{1,1,1}y^3 = 0,$$

where

$$\beta_{i,j,k} = \alpha_{i,j,k}^1 - \alpha_{i,j,k}^2 \quad \text{for } i, j, k \in \{1, 2\}.$$

3. Three-player three-strategy asymmetric game ( $d = 3, n = 3$ ), with the following payoff matrix

Strategy \ Opposing	2, 2	2, 3	3, 2	3, 3	1, 2	1, 3	2, 1	3, 1	1, 1
	1	$\alpha_{2,2}^1$	$\alpha_{2,3}^1$	$\alpha_{3,2}^1$	$\alpha_{3,3}^1$	$\alpha_{1,2}^1$	$\alpha_{1,3}^1$	$\alpha_{2,1}^1$	$\alpha_{3,1}^1$
2	$\alpha_{2,2}^2$	$\alpha_{2,3}^2$	$\alpha_{3,2}^2$	$\alpha_{3,3}^2$	$\alpha_{1,2}^2$	$\alpha_{1,3}^2$	$\alpha_{2,1}^2$	$\alpha_{3,1}^2$	$\alpha_{1,1}^2$
3	$\alpha_{2,2}^3$	$\alpha_{2,3}^3$	$\alpha_{3,2}^3$	$\alpha_{3,3}^3$	$\alpha_{1,2}^3$	$\alpha_{1,3}^3$	$\alpha_{2,1}^3$	$\alpha_{3,1}^3$	$\alpha_{1,1}^3$

The system (8) for three-player three-strategy games is

$$\begin{aligned} \beta_{2,2}^1 y_2^2 + (\beta_{2,3}^1 + \beta_{3,2}^1) y_2 + \beta_{3,3}^1 + (\beta_{1,2}^1 + \beta_{1,3}^1) y_1 y_2 + (\beta_{1,3}^1 + \beta_{3,1}^1) y_1 + \beta_{1,1}^1 y_1^2 &= 0, \\ \beta_{2,2}^2 y_2^2 + (\beta_{2,3}^2 + \beta_{3,2}^2) y_2 + \beta_{3,3}^2 + (\beta_{1,2}^2 + \beta_{1,3}^2) y_1 y_2 + (\beta_{1,3}^2 + \beta_{3,1}^2) y_1 + \beta_{1,1}^2 y_1^2 &= 0, \end{aligned}$$

where

$$\beta_{i,j}^1 = \alpha_{i,j}^1 - \alpha_{i,j}^3, \quad \beta_{i,j}^2 = \alpha_{i,j}^2 - \alpha_{i,j}^3 \quad \text{for } i, j \in \{1, 2, 3\}.$$

**Remark 2.2.** In Section 2.4 we assumed that  $\{\beta_{i_1, \dots, i_{d-1}}^i, \{i_1, \dots, i_{d-1}\} \in \mathcal{A}_{k_1, \dots, k_n}\}$  are iid. We call this condition (A). Recalling from (7) that  $\beta_{i_1, \dots, i_{d-1}}^i := \alpha_{i_1, \dots, i_{d-1}}^i - \alpha_{i_1, \dots, i_{d-1}}^n$ , where  $\alpha_{i_1, \dots, i_{d-1}}^i$  are the payoff entries. It would be more biologically interesting to assume that  $\{\alpha_{i_1, \dots, i_{d-1}}^i, \{i_1, \dots, i_{d-1}\} \in \mathcal{A}_{k_1, \dots, k_n}\}$  are iid. We call this condition (B). Under Condition (B), Condition (A) clearly holds for  $n = 2$ . For  $n > 2$ , it holds only under quite restrictive conditions such as  $\alpha_{k_1, \dots, k_n}^n$  is deterministic or  $\alpha_{k_1, \dots, k_n}^i$  are essentially identical. It is a challenging open problem to work under the general condition (B) for  $n > 2$ .  $\square$

## 3 Statistics of the number of internal equilibria

### 3.1 The expected number of internal equilibria

**Theorem 3.1** (The expected number of internal equilibria). *Suppose that  $\{\beta_{k_1, \dots, k_{n-1}}^i\}$  are iid standard Gaussian random variables. Then the expected number of internal equilibria is*

$$\mathbb{E}(\mathcal{N}_{d,n}) = \frac{1}{2^{n-1}} (d-1)^{\frac{n-1}{2}}. \quad (15)$$

*Proof.* The statement follows directly from Lemma 2.1 and [Kos93, Theorem 3.3 & Corollary 3.4] (see also [EK95, SS93]).  $\square$

### 3.2 The variance of the number of internal equilibria

**Theorem 3.2** (Asymptotic formula for the variance of the number of internal equilibria). *Suppose that  $\{\beta_{k_1, \dots, k_{n-1}}^i\}$  are iid standard Gaussian random variables. Then it holds that*

$$\lim_{d \rightarrow \infty} \frac{4^{n-1} \text{Var}(\mathcal{N}_{d,n})}{(d-1)^{\frac{n-1}{2}}} = V_\infty^2, \quad (16)$$

where  $0 < V_\infty < \infty$  is an explicit constant. Furthermore,  $\mathcal{N}_{d,n}$  satisfies a central limit theorem, that is

$$\frac{4^{n-1} \mathcal{N}_{d,n} - (d-1)^{\frac{n-1}{2}}}{(d-1)^{\frac{n-1}{4}}} \quad (17)$$

converges in distribution, as  $d \rightarrow \infty$ , to a normal random variable with positive variance.

*Proof.* The asymptotic of the variance and the central limit theorem of  $\mathcal{N}$  follow directly from Lemma 2.1 and [AADL18] and [AADL21], respectively (see also [Dal15]).  $\square$

### 3.3 The distribution of the number of internal equilibria for $d$ -player two-strategy games

We provide an analytical formula for the probability that a  $d$ -player two-strategy asymmetric evolutionary game has a certain number of internal equilibria. We use the following notations for the elementary symmetric polynomials

$$\begin{aligned} \sigma_0(y_1, \dots, y_n) &= 1, \\ \sigma_1(y_1, \dots, y_n) &= y_1 + \dots + y_n, \\ \sigma_2(y_1, \dots, y_n) &= y_1 y_2 + \dots + y_{n-1} y_n \\ &\vdots \\ \sigma_{n-1}(y_1, \dots, y_n) &= y_1 y_2 \dots y_{n-1} + \dots + y_2 y_3 \dots y_n, \\ \sigma_n(y_1, \dots, y_n) &= y_1 \dots y_n; \end{aligned}$$

and denote

$$\Delta(y_1, \dots, y_n) = \prod_{1 \leq i < j \leq n} |y_i - y_j|$$

the Vandermonde determinant. The main result of this section is the following theorem

**Theorem 3.3.** *Suppose that the random variables  $b_0, b_1, \dots, b_{d-1}$  defined in (13) have a joint density  $p(a_0, \dots, a_{d-1})$ . Then the probability that a  $d$ -player two-strategy asymmetric random evolutionary game has  $m$  ( $0 \leq m \leq d-1$ ) internal equilibria is*

$$p_m = \sum_{k=0}^{\lfloor \frac{d-1-m}{2} \rfloor} p_{m, 2k, d-1-m-2k},$$

where  $p_{m, 2k, d-1-m-2k}$  is given by

$$\begin{aligned} p_{m, 2k, d-1-m-2k} &= \frac{2^k}{m! k! (d-1-m-2k)!} \int_{\mathbf{R}_+^m} \int_{\mathbf{R}_+^{d-1-2k-m}} \int_{\mathbf{R}_+^k} \int_{[0, \pi]^k} \int_{\mathbf{R}} \\ &\quad r_1 \dots r_k p(a\sigma_0, \dots, a\sigma_{d-1}) |a|^{d-1} \Delta da d\alpha_1 \dots d\alpha_k dr_1 \dots dr_k dx_1 \dots dx_{d-1-2k}. \end{aligned}$$

When  $\{\beta_{i_1, \dots, i_{d-1}}\}$  are iid normal Gaussian random variables,  $p_{m, 2k, d-1-m-2k}$  can be expressed as

$$p_{m, 2k, d-1-m-2k} = \frac{2^k}{m!k!(d-1-m-2k)!} \frac{\Gamma\left(\frac{d}{2}\right)}{(\pi)^{\frac{d}{2}} \prod_{i=0}^{d-1} \delta_i^{\frac{1}{2}}} \int_{\mathbf{R}_+^m} \int_{\mathbf{R}_+^{d-1-2k-m}} \int_{\mathbf{R}_+^k} \int_{[0, \pi]^k} r_1 \dots r_k$$

$$\left(\sum_{i=0}^{d-1} \frac{\sigma_i^2}{\delta_i}\right)^{-\frac{d}{2}} \Delta d\alpha_1 \dots d\alpha_k dr_1 \dots dr_k dx_1 \dots dx_{d-1-2k}.$$

In the above formula,  $\delta_i = \binom{d-1}{i}$  and  $\sigma_i$ , for  $i = 0, \dots, d-1$ , and  $\Delta$  are given by

$$\sigma_j = \sigma_j(x_1, \dots, x_{n-2k}, r_1 e^{i\alpha_1}, r_1 e^{-i\alpha_1}, \dots, r_k e^{i\alpha_k}, r_k e^{-i\alpha_k}),$$

$$\Delta = \Delta(x_1, \dots, x_{n-2k}, r_1 e^{i\alpha_1}, r_1 e^{-i\alpha_1}, \dots, r_k e^{i\alpha_k}, r_k e^{-i\alpha_k}).$$

In particular, the probability that a  $d$ -player two-strategy random evolutionary game has the maximal number of internal equilibria is:

$$p_{d-1} = \frac{1}{(d-1)!} \frac{\Gamma\left(\frac{d}{2}\right)}{(\pi)^{\frac{d}{2}} \prod_{i=0}^{d-1} \delta_i^{\frac{1}{2}}} \int_{\mathbf{R}_+^{d-1}} \left(\sum_{i=0}^{d-1} \frac{\sigma_i^2(x_1, \dots, x_{d-1})}{\delta_i}\right)^{-\frac{d}{2}} \Delta(x_1, \dots, x_{d-1}) dx_1 \dots dx_{d-1}.$$

*Proof.* The proof of this theorem follows the same approach as that of [DTH19, Theorem 4], and we omit it here.  $\square$

In Figure 1 we compute the probability of having a certain number of internal equilibria for some small games using the analytical formulae given in Theorem 3.3 and compare it with results from extensive numerical simulation by sampling the payoff matrix entries.

### 3.4 Universality phenomena

In Sections 3.1 and 3.2, we assume that the random coefficients  $\beta_i$  are standard normal distributions. Direct applications of recent results in random polynomial theory allow us to remove this assumption, obtaining universality phenomena that characterize the asymptotic behaviour of the expected value and variance of the number of internal equilibria for  $d$ -player two-strategy games for a large class of general distributions. We recall from Section 2.5 that finding an internal equilibrium for a  $d$ -player two-strategy random asymmetric evolutionary game amounts to finding a positive root of the random polynomial (12) with coefficients  $b_k$  determined from the payoff entries via (13). From this formula, suppose that  $\beta_{i_1, \dots, i_{d-1}}$  are iid random variable, then  $b_k$  is again a centered random variable with variance  $\binom{d-1}{k}$ . Thus, we can write  $b_k$  as

$$b_k = \sqrt{\binom{d-1}{k}} \xi_k, \tag{18}$$

where  $\xi_k$  is a centered random variable with variance 1.

**Theorem 3.4** (Universality for the expected number of internal equilibria). *Suppose that the random variables  $\{\xi_k\}$  are independent with mean 0, variance 1 and finite  $(2 + \varepsilon)$ -moment for some  $\varepsilon > 0$ . Then*

$$\mathbb{E}(\mathcal{N}_{d,2}) = \frac{\sqrt{d-1}}{2} + O((d-1)^{1/2-c}),$$

for some  $c > 0$  depending only on  $\varepsilon$ .



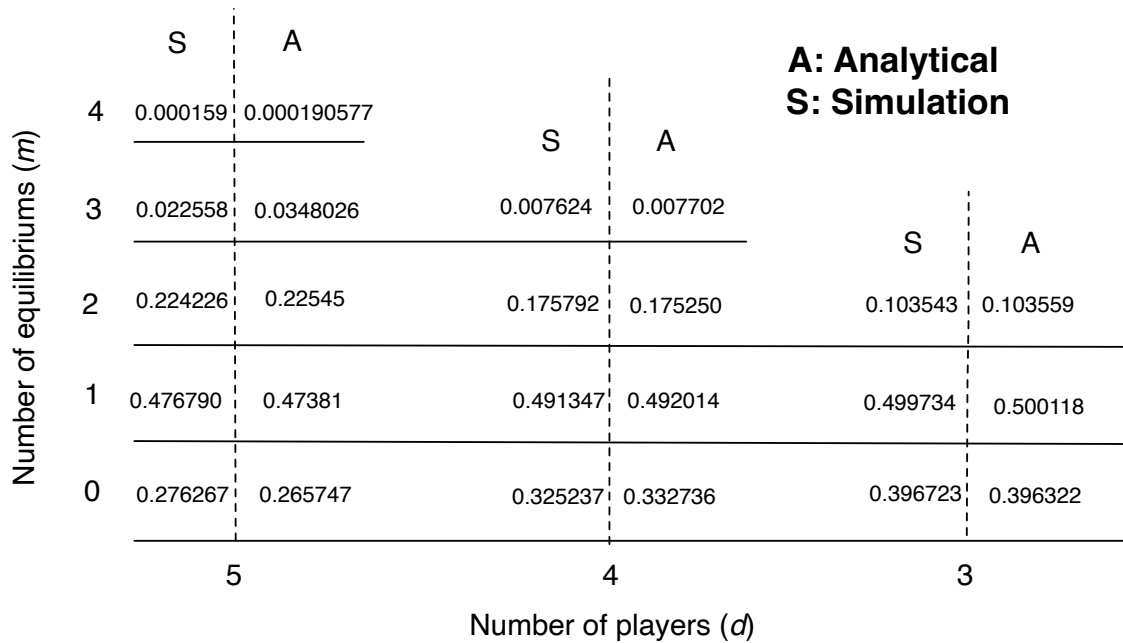


Figure 1: Numerical calculations versus simulations of the probability of having a concrete number ( $m$ ) of internal equilibria,  $p_m$ , for different values of  $d$ . Analytical results are obtained from analytical formulas (Theorem 3). Simulation results are obtained based on sampling  $10^6$  payoff matrices. Analytical and simulations results are closely in accordance with each other. All results are obtained using Mathematica.

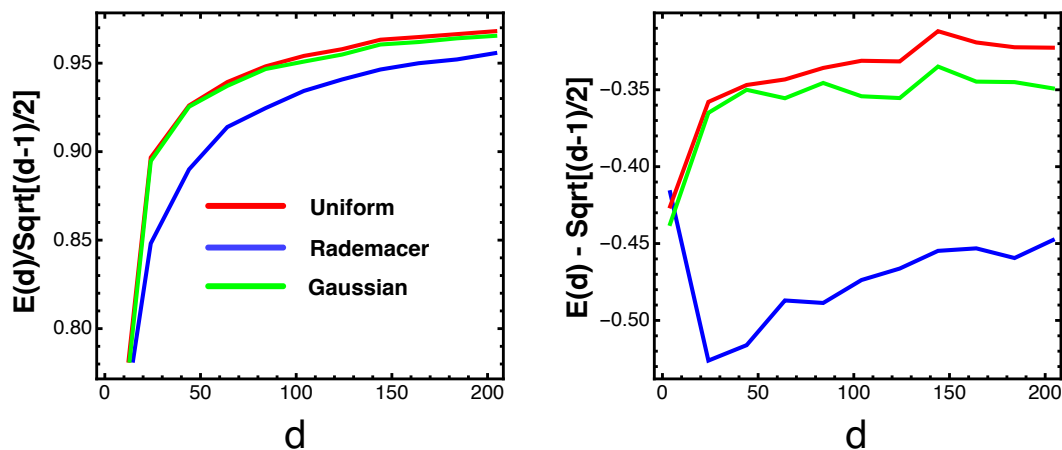


Figure 2: Universality phenomena. Simulation results for varying  $d$  for different distributions (Gaussian, Uniform, Rademacer) corroborate analytical results. All simulation results are obtained based on sampling  $10^6$  payoff matrices from the corresponding distributions. All results are obtained using Mathematica.

*Proof.* This is a direct consequence of Lemma 2.1 and [TV15] (see also [BD04, FK20, NV22], in particular [NV22] for a stronger statement where the assumptions on the random variables  $\{\xi_k\}$  are relaxed).  $\square$

## 4 Symmetric vs asymmetric evolutionary games

In previous works [DH15, DH16, DTH19, CDP22], we studied the statistics of the number of internal equilibria for  $d$ -player two-strategy random *symmetric* evolutionary games, in which the payoff of a player in a group interaction is independent of the ordering of its members. In this symmetric case, instead of (12) with coefficients given by (18), we obtain a different class of random polynomial

$$P^{\text{sym}}(y) = \sum_{k=0}^{d-1} \binom{d-1}{k} \xi_k y^k.$$

Let  $\mathcal{N}_{d,2}^{\text{sym}}$  be the number of internal equilibria for  $d$ -player two-strategy symmetric games when the coefficient  $\xi_k$  are iid standard Gaussian random variables. Then [CDP22] establishes a lower bound for the expected value of  $\mathcal{N}_{d,2}^{\text{sym}}$  for all  $d$ ,

$$\mathbb{E}(\mathcal{N}_{d,2}^{\text{sym}}) \geq \frac{\sqrt{d-1}}{2} \quad \text{for all } d > 1, \quad (19)$$

and its asymptotic behaviour as  $d \rightarrow +\infty$

$$\mathbb{E}(\mathcal{N}_{d,2}^{\text{sym}}) = \sqrt{\frac{d-1}{2}}(1 + o(1)) \quad \text{as } d \rightarrow +\infty. \quad (20)$$

The lower bound (19) is *precisely* the expected number of internal equilibria for  $d$ -player two-strategy asymmetric games obtained in (15). This has an interesting biological interpretation: symmetry increases the expected number of internal equilibria (and hence, the biological or behavioural diversity).

In Figure 2 we numerically compute the asymptotic behaviour of  $\mathbb{E}(\mathcal{N}_{d,2}^{\text{sym}})$  for three most popular classes of distribution

- (i)  $\xi_i$  are iid standard Gaussian distributions,
- (ii)  $\xi_i$  are iid Rademacher distributions (i.e., receiving discrete values either  $+1$  or  $-1$  with equal probability  $1/2$ ),
- (iii)  $\{\xi\}_i$  are uniformly distributed on  $[-1, 1]$ .

We observe that, as  $d \rightarrow +\infty$ , the leading order of  $\mathbb{E}(\mathcal{N}_{d,2}^{\text{sym}})$  is the same, which is  $\sqrt{\frac{d-1}{2}}$  as in (20), in all cases; while the next order term is uniformly bounded but with different bounds for different distributions. This is similar to the elliptic random polynomials arising from asymmetric games [DNV15]. We conjecture that universality phenomenon and central limit theorem also hold true for symmetric games.

## 5 Conclusion and discussion

In summary, we have established an appealing connection between evolutionary game theory and random polynomial theory: the class of random polynomials arising from the study of equilibria of random asymmetric evolutionary games is exactly the celebrated Kostlan-Shub-Smale system of random polynomials. The connection has enabled us to immediately obtain the statistics of the number of internal equilibria of random asymmetric evolutionary games. As a consequence

of our analysis, we have also proved that symmetry increases biological diversity. Furthermore, we have also numerically observed universality properties for the number of internal equilibria of symmetric random evolutionary games. Rigorously proving the universality phenomenon and a central limit theorem for symmetric games is a challenging open problem for future work. Our work also opens the door for further discoveries on the links between the two well-established theories, of evolutionary game theory and of random polynomials, for more complicated dynamics, such as the replicator-mutator equation.

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