

## Revision Notes (#Ref. No.: Access-2019-01124)

# "An Improved Complex-valued Recurrent Neural Network Model for Time-Varying Complex-valued Sylvester Equation"

Lei Ding, Lin Xiao, Kaiqing Zhou, Yonghong Lan, Yongsheng Zhang, and Jichun Li

First of all, we (the authors) would like to express our sincere gratitude to the editors and anonymous reviewers for their time, effort and recognition given to our manuscript Access-2019-01124 entitled "An Improved Complex-valued Recurrent Neural Network Model for Time-Varying Complex-valued Sylvester Equation". For example, "Two reviewers have been obtained on this paper, and all are positive", "... The results shows that the sign-multi-power activation function outperforms the sign-bi-power activation function and the IZNN activation function" (Reviewer 1), "The simulation results show that compared with the sign-bi-power model and the IZNN model, the ICZNN model increases the convergence speed about percent 34% and 62%, respectively. So this is a meaningful work" (Reviewer 2). Thanks a lot and best regards.

Secondly, it is worth pointing out that the editors' and reviewers' comments and suggestions have really helped us improve the quality and presentation of our manuscript further. In light of their inspiring comments and suggestions, we have revised our manuscript duly and carefully, with the main revisions are listed as follows.

1) As suggested by Reviewer 1, After the formula (11), the symbol ‘;’ in the sentence of ‘where  $q>0;...z>1;q>0;...$ ’ have been revised as ‘,’.

2) As suggested by Reviewer 1 and Reviewer 2, we choose 3 randomly generated initial value to show the convergence property.

3) As suggested by Reviewer 1, the word ‘Fig.’ in the sentence ‘the .....are displayed in Fig.1-15. The Fig. 1-4, the Fig. 6-9 and the Fig. 11-14 display....’ have been revised as ‘Figs.’.

4) As suggested by Reviewer 2, the word ‘can’t’ in the ‘which causes this ZNN model obtain its ideal solution in finite time’ has been revised as ‘cannot’.

5)As suggested by Reviewer 2, the sentence ‘In [35] Ding et al. designed ..... (CVLE), in which a CVLE is transformed ....’ has been revised as ‘In [35] Ding et al. designed ..... (CVLE), which is transformed ....’.

## **Responses to Comments of Associate Editor (AE)**

**Comments.** *“Two reviewers have been obtained on this paper, and all are positive. However, this paper needs some revisions before it can be considered for publication.”*

**Response.** Thank the AE for his time, effort and recognition given to our manuscript. As suggested by two reviewers, the authors have revised the manuscript over and over carefully to improve the presentation and quality of the manuscript. We hope that we can present a high quality paper in the journal. Thanks again.

### **Response to Comments of Reviewer 1**

**Comment 1.** *“After the formula (11), the symbol ‘;’ in the sentence of ‘where  $q>0;...z>1;q>0;...$ ’ should be ‘,’”.*

**Response.** Thank the reviewer sincerely for pointing out such a comment. As suggested, in the revised manuscript, the symbol ‘;’ in the sentence of ‘where  $q>0;...z>1;q>0;...$ ’ have been revised as ‘,’”.

**Comment 2.** *“In Section IV, there should be more than one output trajectories of neural states in the corresponding figures to show the convergence property when different initial values are generated.”*

**Response.** Thank the reviewer sincerely for pointing out such a comment. As suggested, we choose 4 randomly generated initial value, which can generate 4 output trajectories of the neural state, to show the convergence property when different initial values are generated.

**Comment 3.** *“In Section IV, the word ‘Fig.’ in the sentence ‘the .....are displayed in Fig.1-15. The Fig. 1-4, the Fig. 6-9 and the Fig. 11-14 display....’ should be ‘Figs.’”*

**Response.** Thank the reviewer sincerely for pointing out such a comment. As suggested, in the revised manuscript, the word ‘Fig.’ in the sentence ‘the .....are displayed in Fig.1-15. The Fig. 1-4, the Fig. 6-9 and the Fig. 11-14 display....’ has been revised as Figs.’

## Response to Comments of Reviewer 2

**Comment 1.** *“In the simulation process, only one randomly generated initial value is given, I advise that there should be more randomly generated initial values are given, for example 3 or 4 randomly generated initial values should be used in each simulation process.”*

**Response.** Thank the reviewer sincerely for pointing out such a comment. As suggested, we choose 4 randomly generated initial value in each simulation process.

**Comment 2.** *“In the Section I, the word ‘can’t’ in the ‘which causes this ZNN model obtain its ideal solution in finite time’ should be represented as ‘cannot’”*.

**Response.** Thank the reviewer sincerely for pointing out such a comment. As suggested, in the revised manuscript, the word ‘can’t’ in the ‘which causes this ZNN model obtain its ideal solution in finite time’ have been revised as ‘cannot’.

**Comment 3.** *“In the Section I, the sentence ‘In [35] Ding et al. designed ..... (CVLE), in which a CVLE is transformed ....’ should be represented as ‘In [35] Ding et al. designed ..... (CVLE), which is transformed ....’”*.

**Response.** Thank the reviewer sincerely for pointing out such a comment. As suggested, the sentence In [35] Ding et al. designed ..... (CVLE), in which a CVLE is transformed ....’ has been revised as ‘In [35] Ding et al. designed ..... (CVLE), which is transformed ....’ .

Finally, we would like to say thanks again sincerely to the reviewers for their time and effort spent in reviewing the manuscript, as well as the constructive comments which have inspired the authors to think about and improve further the presentation and quality of this manuscript.

# An Improved Complex-valued Recurrent Neural Network Model for Time-Varying Complex-valued Sylvester Equation

Lei Ding, Lin Xiao, Kaiqing Zhou, Yonghong Lan, Yongsheng Zhang, and Jichun Li

**Abstract**—Complex-valued time-varying Sylvester equation (CVTVSE) has been successfully applied into mathematics and control domain. However the computation load of solving CVTVSE will rise significantly with the increase of sampling rate, and it is a challenging job to tackle online the CVTVSE. In this paper, a new sign-multi-power activation function is designed. Based on this new activation function, an improved complex-valued Zhang neural network (ICZNN) model for tackling the CVTVSE is established. Furthermore, the strict proof for the maximum time of global convergence of the ICZNN is given in detail. Two numerical experiments are employed to verify the performance of the proposed ICZNN model, and the results show that, as compared with the previous Zhang neural network (ZNN) models with different nonlinear activation functions, this ICZNN model with sign-multi-power activation function has a faster convergence speed to tackle the CVTVSE.

**Index Terms**—Zhang neural network, complex-valued time-varying Sylvester equation, convergence speed, sign-multi-power function, finite-time convergence.

## I. INTRODUCTION

TODAY the Sylvester equation (SE) has been successfully applied into many fields, such as the robotic application [1], the waveguide eigenvalue problem [2], the commutative rings [3], the isogeometric Preconditioners [4], the multi-agent linear parameter-varying systems [5]. Generally speaking, the Sylvester equation can be divided into two categories: namely the static SE and the dynamic SE (i.e., time-varying Sylvester equation, TVSE). The classical algorithms to tackle the static SE are the Bartels–Stewart and Hessenberg–Schur methods [6], [7]. The main shortcoming of the above algorithms is that they only fit for solving the small-scale problems due to the dense matrix operation. Recently, a series of iteration

algorithms using the gradient information have been proposed to tackle the static SE, such as the relaxed gradient based iterative algorithm [8], the least-squares iterative algorithm [9], the accelerated gradient algorithm [10], and the alternating direction implicit algorithm [11]. However, the above methods can not effectively tackle the TVSE online. The main reason is that the TVSE should be calculated in every sampling cycle, and the computational burden will significantly increase within a sampling cycle when the sampling rate increases. Thus the above algorithm may not complete a calculation if the computational burden is too big. Today the neural networks have caused widely attention [12]–[14]. As a kind of neural network, the recurrent neural networks (RNNs) have a stronger real-time computation ability than the traditional numerical algorithms [15]–[21]. So a series of RNN models have been designed for tackling the dynamic SE. For example, the gradient-based RNNs are designed to tackle the real-valued SE [22], [23]. But the gradient-based RNNs may need very long time to obtain its ideal solution because its performance indicator is the Frobenius norm of errors. So a novel RNN called Zhang neural network (ZNN), which can converge to zero exponentially, is proposed because its performance indicator is a vector/matrix-valued error function [24]–[30]. But Xiao pointed out that the traditional ZNN cannot obtain its theoretical solution in finite time [31]. So a series of improved ZNNs with the finite time convergence property have been proposed [32]–[34]. Furthermore, some of improved ZNNs have been successfully employed to tackle the TVSE online [35]–[37].

Now the complex-valued neural networks have shown more advantages than the real-valued neural networks in some fields, such as the high-capacity auto-associative memories [38], the spectral domain [39], the millimeter-wave active imaging [40], and the geometric measures [41]. Inspired by the previous studies for the ZNNs, we explore a novel complex-valued ZNN model for solving the complex-valued time-varying Sylvester equation (CVTVSE) in this paper. Before that, some related work about complex-valued ZNNs is reviewed as follows. In [42], a ZNN model is applied to tackle a complex matrix inversion. However, a linear activation function is used in this ZNN model, which causes this ZNN model cannot obtain its ideal solution in finite time. In [43] Li et al. proposed a novel sign-bi-power nonlinear activation function to build an improved ZNN model, which can obtain its theoretical results in finite time for tackling the TVSE. We can describe this

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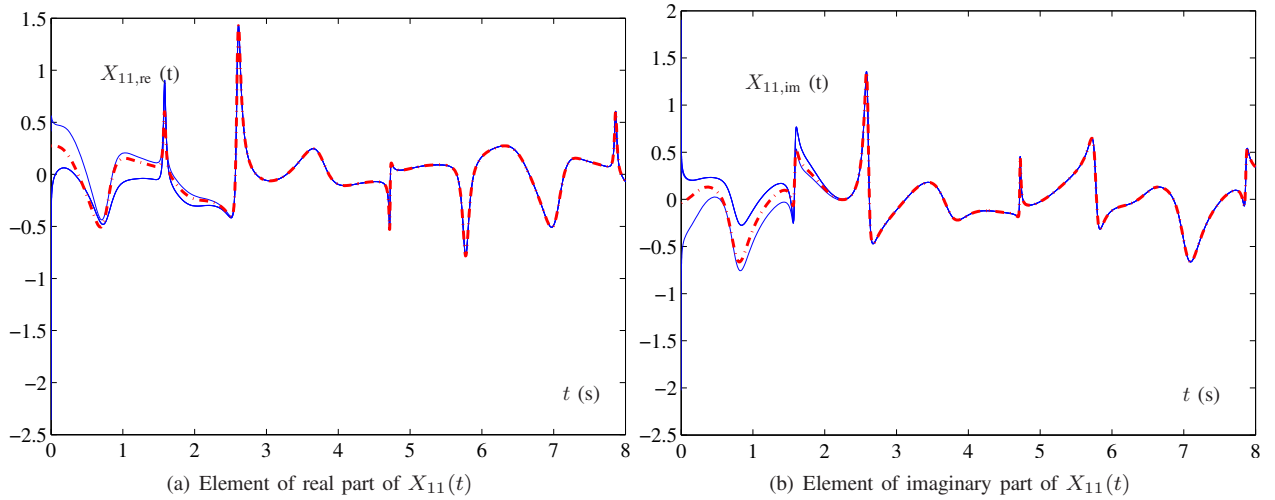


Fig. 1. Output trajectories of neural states  $X_{11}(t)$  synthesized by the model (9) in example 1. The dotted red line denotes the theoretical values, and the blue solid line denotes the calculated values.

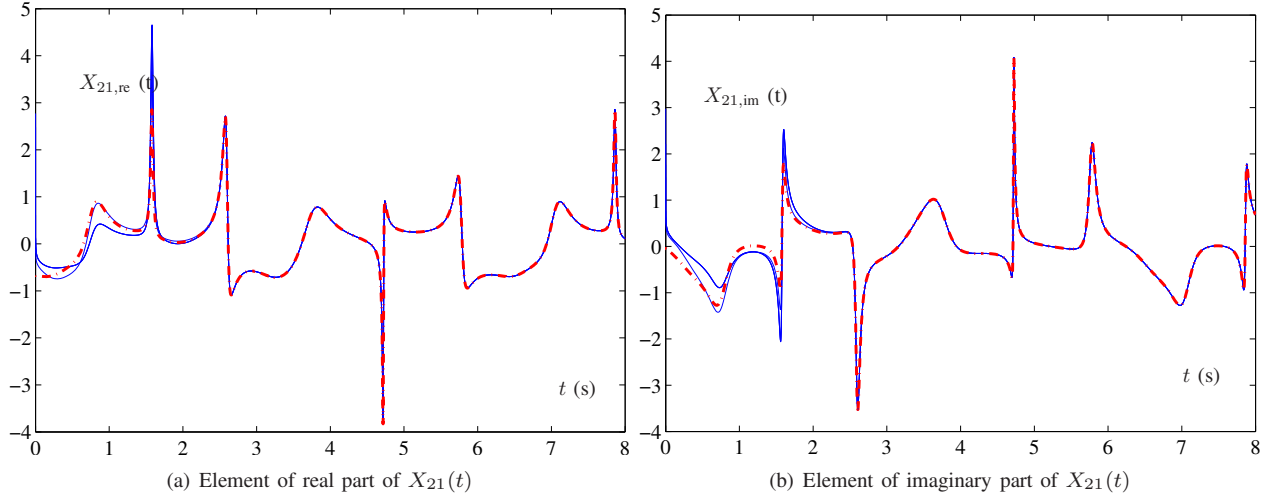


Fig. 2. Output trajectories of neural states  $X_{21}(t)$  synthesized by the model (9) in example 1. The dotted red line denotes the theoretical values, and the blue solid line denotes the calculated values.

sign-bi-power function as

$$\Psi(u) = \frac{1}{2} \text{sgn}^z(u) + \frac{1}{2} \text{sgn}^{1/z}(u), \quad (1)$$

where  $z$  is an odd integer and satisfy  $z > 1$ , and

$$\text{sgn}^z(u) = \begin{cases} |u|^z, & \text{if } u > 0 \\ 0, & \text{if } u = 0 \\ -|u|^z, & \text{if } u < 0. \end{cases}$$

Furthermore, in [44] Li et al. proposed a complex-valued ZNN based on the sign-bi-power function to tackle the CVTVSE. Inspired by the sign-bi-power function, a tunable activation function is designed to obtain a higher convergence rate in [45]. In [46] Ding et al. designed an improved ZNN activation function to tackle the complex-valued linear equations (CVLE), which is transformed into a real-valued linear equation, and the improved ZNN activation function can be described as

$$\Psi(u) = \text{sign}(u)(j_1|u|^h + j_2|u|^{1/h} - j_3|u|), \quad (2)$$

where  $j_1 > j_3 > 0$ ,  $j_2 > j_3 > 0$ ,  $h$  is an odd integer and satisfies  $h > 1$ , and

$$\text{sign}(u) = \begin{cases} 1, & \text{if } u > 0 \\ 0, & \text{if } u = 0 \\ -1, & \text{if } u < 0. \end{cases}$$

According to the above idea, to obtain a higher convergence rate for online solving the CVTVSE, an improved nonlinear activation function is designed and investigated in this paper. Based on this new activation function, an improved complex-valued Zhang neural network (ICZNN) model for tackling the CVTVSE is established. Furthermore, the strict proof for the maximum time of global convergence of the ICZNN is given in detail. Two numerical experiments are employed to verify the performance of the proposed ICZNN model.

The remaining parts contain the following content. In Section II, we give the description of the problem. In Section III, we design a sign-multi-power function to build a novel ICZNN to tackle the CVTVSE, and give the theoretical proof

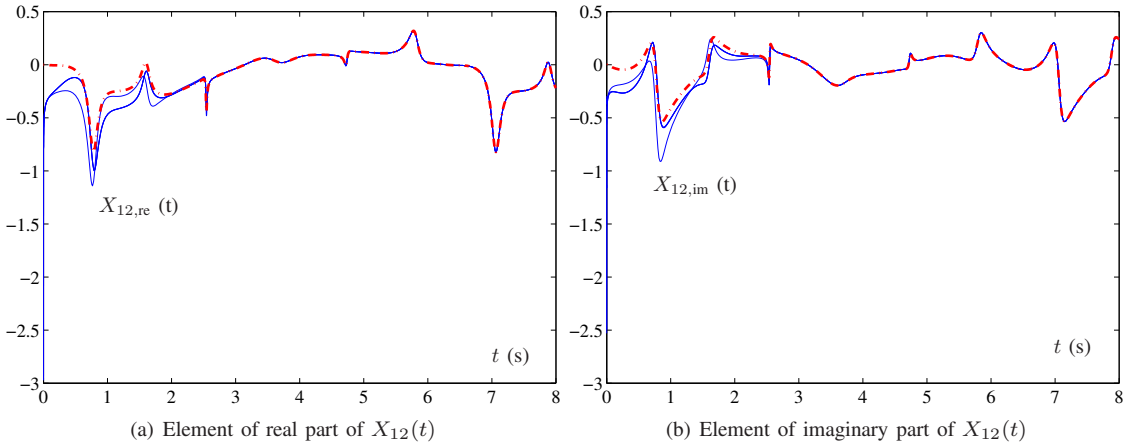


Fig. 3. Output trajectories of neural states  $X_{12}(t)$  synthesized by the model (9) in example 1. The dotted red line denotes the theoretical values, and the blue solid line denotes the calculated values.

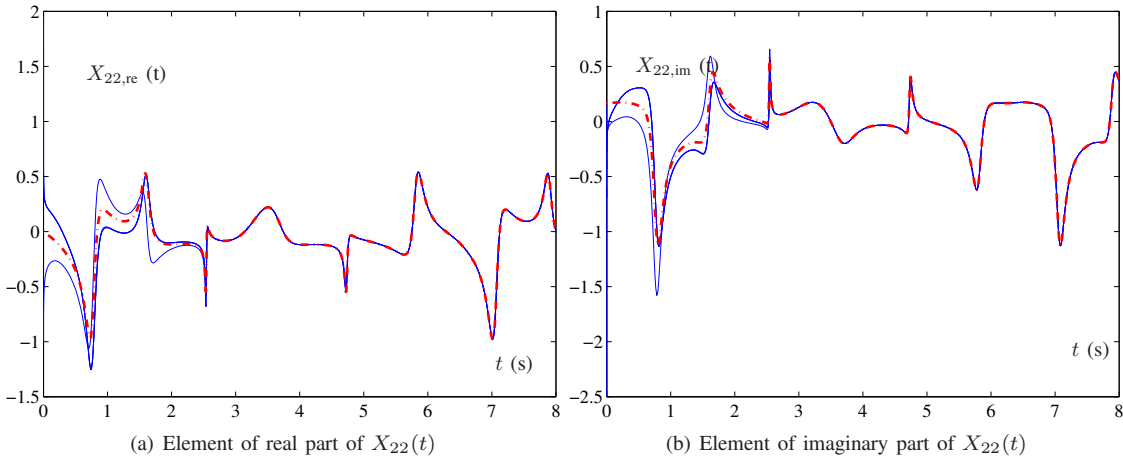


Fig. 4. Output trajectories of neural states  $X_{22}(t)$  synthesized by the model (9) in example 1. The dotted red line denotes the theoretical values, and the blue solid line denotes the calculated values.

for the maximum time of global convergence of the ICZNN. In Section IV, we give two digit experiments to verify the superiority of the sign-multi-power function. Finally, we give the final conclusions of this paper in Section V.

Before finishing this section, we can summarize the contribution of this paper as below.

- 1) A novel sign-multi-power activation function is designed.
- 2) A novel ICZNN is derived to tackle the CVTVSE in complex-valued domain, and the strict theoretical proof is explained.
- 3) The digit experiments demonstrate that this novel model for online tackling the CVTVSE can increase the convergence rate significantly.

## II. DESCRIPTION OF THE PROBLEM

The CVTVSE can be described as

$$G(t)X(t) - X(t)Q(t) = -S(t) \in \mathbb{C}^{n \times n}, \quad (3)$$

where  $G(t)$ ,  $Q(t)$  and  $S(t)$  are all the complex-valued coefficient matrices,  $t$  means time, and  $X(t)$  is a time-varying matrix needs to be calculated. Now we give the following assumptions: the complex-valued matrices  $G(t)$ ,  $Q(t)$ , and  $S(t)$  have no

identical eigenvalues, and are all first-order differentiable. So there will be only a solution for the equation (3). To help the future description, let  $\tilde{X}(t)$  denote the theoretical solution. Our target is to design a novel nonlinear complex-valued activation function to build a neural network for tackling the CVTVSE.

First suppose  $G(t)$ ,  $Q(t)$ ,  $X(t)$  and  $S(t)$  are all real-valued matrices, and the procedure for the real-valued TVSE using the ZNN model can be described as the following three steps.

Step 1: The error function can be represented as:

$$D(t) = G(t)X(t) - X(t)Q(t) + S(t). \quad (4)$$

Step 2: The evolution procedure is designed as follows:

$$\dot{D}(t) = q\Psi(D(t)), \quad (5)$$

where  $q > 0$  denotes the coefficient to accelerate the convergence rate, and  $\Psi(\cdot)$  denotes the activation function.

Step 3: Substitute (4) into (5), and we will have the following equation:

$$G(t)\dot{X}(t) - \dot{X}(t)Q(t) = q\Psi(G(t)X(t) - X(t)Q(t) + S(t)) - \dot{G}(t)X(t) + X(t)\dot{Q}(t) - \dot{S}(t). \quad (6)$$

Now suppose  $G(t) = G_{\text{re}}(t) + jG_{\text{im}}(t)$ ,  $Q(t) = Q_{\text{re}}(t) + jQ_{\text{im}}(t)$ ,  $X(t) = X_{\text{re}}(t) + jX_{\text{im}}(t)$  and  $S(t) = S_{\text{re}}(t) + jS_{\text{im}}(t)$ ,

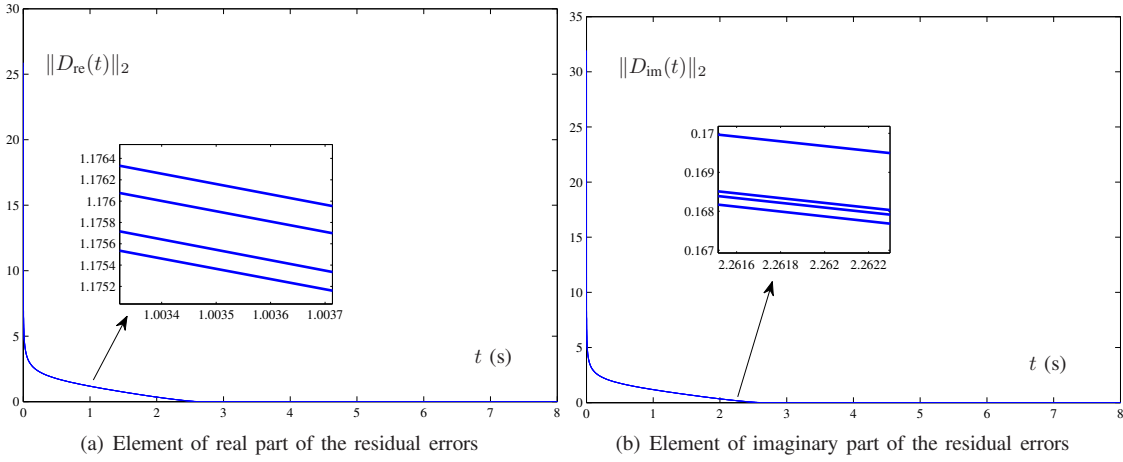


Fig. 5. Output trajectories of the residual errors synthesized by the model (10) in example 1.

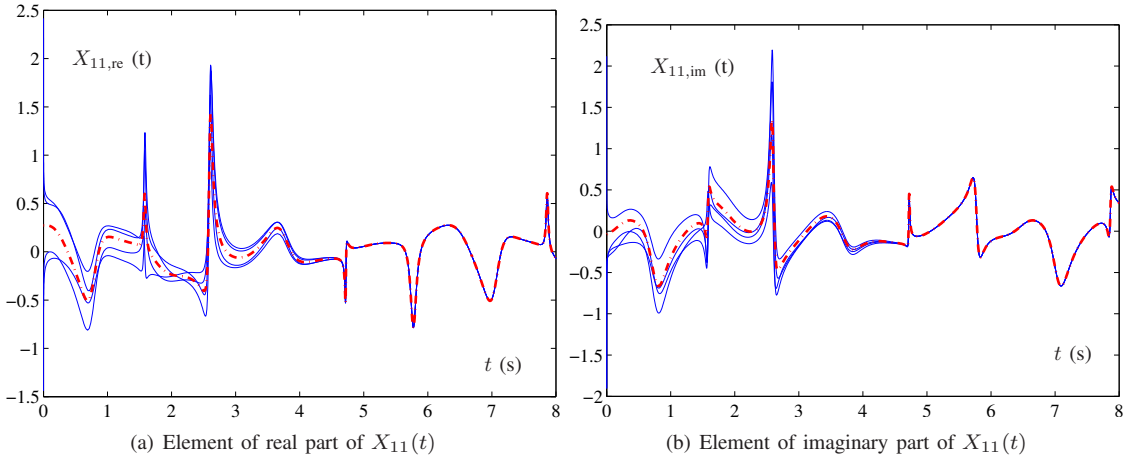


Fig. 6. Output trajectories of neural states  $X_{11}(t)$  synthesized by the model (10) in example 1. The dotted red line denotes the theoretical values, and the blue solid line denotes the calculated values.

where  $j = \sqrt{-1}$  denotes an imaginary unit. Then according to the equation (6), we have

$$\begin{aligned}
 G(t)\dot{X}(t) - \dot{X}(t)Q(t) &= q(\Psi(G_{re}(t)X_{re}(t) \\
 &\quad - G_{im}(t)X_{im}(t) - X_{re}(t)Q_{re}(t) \\
 &\quad + X_{im}(t)Q_{im}(t)) + j\Psi(G_{im}(t)X_{re}(t) \\
 &\quad + G_{re}(t)X_{im}(t) - X_{im}(t)Q_{re}(t) \\
 &\quad - X_{re}(t)Q_{im}(t))) - \dot{G}(t)X(t) \\
 &\quad + X(t)\dot{Q}(t) - \dot{S}(t).
 \end{aligned} \tag{7}$$

### III. A NOVEL RECURRENT NEURAL NETWORK

#### A. A New Nonlinear Activation Function

From the equation (7), we can find that a suitable activation function will increase the convergence rate significantly. So a novel nonlinear activation function called the sign-multi-power function can be designed as follows:

$$\begin{aligned}
 \Psi(k) &= a_1 \text{sgn}^z(k) + a_2 \text{sgn}^{z-2^1}(k) + a_3 \text{sgn}^{z-2^2}(k) \\
 &\quad + \dots + a_n \text{sgn}^1(k) + a_{n+1} \text{sgn}^{1/z}(k),
 \end{aligned} \tag{8}$$

where  $z$  is odd integer and satisfy  $z > 1$ , the parameters  $a_1, \dots, a_{n+1}$  are all the positive numbers, and

$$\text{sgn}^z(u) = \begin{cases} |u|^z, & \text{if } u > 0 \\ 0, & \text{if } u = 0 \\ -|u|^z, & \text{if } u < 0. \end{cases}$$

#### B. A Sign-multi-power Model for Tackling the CVTVSE

For ease of comparison, we first introduce two improved ZNN models. One is a sign-bi-power model [44], and the other is an IZNN model [46]. For ease of description, we first give the following definition:

$$\begin{aligned}
 f_1(t) &= G_{re}(t)X_{re}(t) - G_{im}(t)X_{im}(t) \\
 &\quad - X_{re}(t)Q_{re}(t) + X_{im}(t)Q_{im}(t),
 \end{aligned}$$

and

$$\begin{aligned}
 f_2(t) &= G_{im}(t)X_{re}(t) + G_{re}(t)X_{im}(t) \\
 &\quad - X_{im}(t)Q_{re}(t) - X_{re}(t)Q_{im}(t).
 \end{aligned}$$

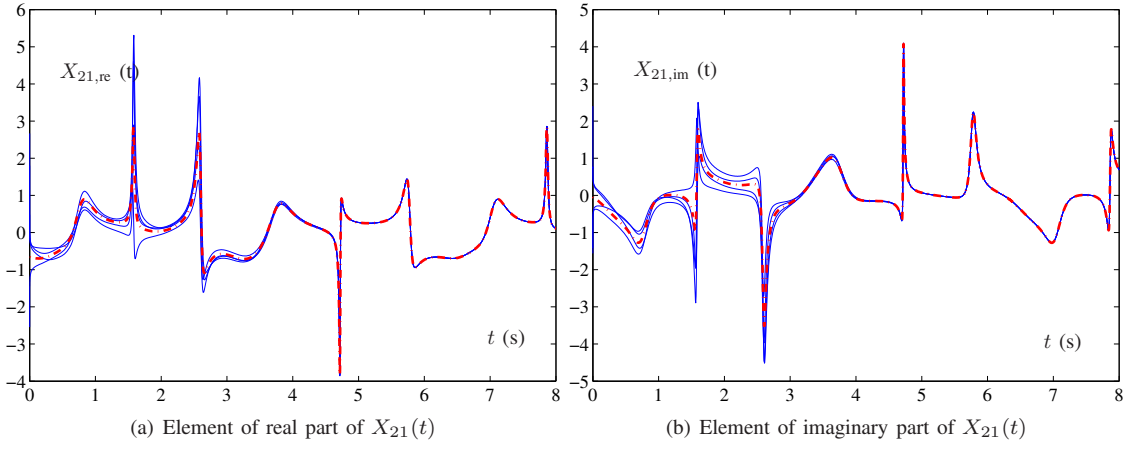


Fig. 7. Output trajectories of neural states  $X_{21}(t)$  synthesized by the model (10) in example 1. The dotted red line denotes the theoretical values, and the blue solid line denotes the calculated values.

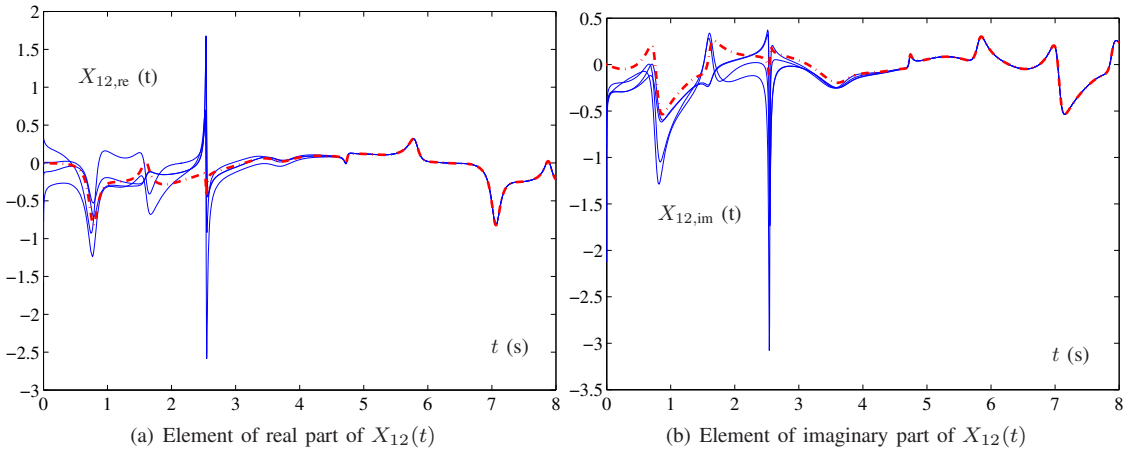


Fig. 8. Output trajectories of neural states  $X_{12}(t)$  synthesized by the model (10) in example 1. The dotted red line denotes the theoretical values, and the blue solid line denotes the calculated values.

Then the sign-bi-power model is represented as:

$$\begin{aligned}
 G(t)\dot{X}(t) - \dot{X}(t)Q(t) &= q\left(\frac{1}{2}\text{sgn}^z(f_1(t))\right. \\
 &+ \frac{1}{2}\text{sgn}^{1/z}(f_1(t)) + j\left(\frac{1}{2}\text{sgn}^z(f_2(t))\right. \\
 &+ \left.\left.\frac{1}{2}\text{sgn}^{1/z}(f_2(t))\right)\right) - \dot{G}(t)X(t) \\
 &+ X(t)\dot{Q}(t) - \dot{S}(t).
 \end{aligned} \quad (9)$$

The IZNN model is represented as:

$$\begin{aligned}
 G(t)\dot{X}(t) - \dot{X}(t)Q(t) &= q(\text{sign}(f_1(t))(j_1|f_1(t)|^h \\
 &+ j_2|f_1(t)|^{1/h} - j_3|f_1(t)|) \\
 &+ j(\text{sign}(f_2(t))(j_1|f_2(t)|^h \\
 &+ j_2|f_2(t)|^{1/h} - j_3|f_2(t)|)) - \dot{G}(t)X(t) \\
 &+ X(t)\dot{Q}(t) - \dot{S}(t),
 \end{aligned} \quad (10)$$

where  $j_1 > j_3 > 0$ ,  $j_2 > j_3 > 0$ , and  $h$  is an odd integer and satisfy  $h > 1$ , and

$$\text{sign}(u) = \begin{cases} 1, & \text{if } u > 0 \\ 0, & \text{if } u = 0 \\ -1, & \text{if } u < 0. \end{cases}$$

Now we can build a novel improved ZNN using the sign-multi-power function, which is designed as:

$$\begin{aligned}
 G(t)\dot{X}(t) - \dot{X}(t)Q(t) &= q(a_1\text{sgn}^z(f_1(t)) \\
 &+ a_2\text{sgn}^{z-2^1}(f_1(t)) + a_3\text{sgn}^{z-2^2}(f_1(t)) \\
 &+ \cdots + a_n\text{sgn}^1(f_1(t)) + a_{n+1}\text{sgn}^{1/z}(f_1(t)) \\
 &+ j(a_1\text{sgn}^z(f_2(t)) + a_2\text{sgn}^{z-2^1}(f_2(t)) \\
 &+ a_3\text{sgn}^{z-2^2}(f_2(t)) + \cdots + a_n\text{sgn}^1(f_2(t)) \\
 &+ a_{n+1}\text{sgn}^{1/z}(f_2(t)))) - \dot{G}(t)X(t) \\
 &+ X(t)\dot{Q}(t) - \dot{S}(t),
 \end{aligned} \quad (11)$$

where  $q > 0$ ,  $z$  is an odd integer and satisfies  $z > 1$ ,  $q > 0$ , and the parameters  $a_1, \dots, a_{n+1}$  are all the positive numbers. We can call this model (11) as the ICZNN model.

### C. Theorem Analysis of ICZNN Model

*Theorem 1:* The ICZNN model (11) is globally stable no matter what its randomly generated initial value  $X(0)$  is.

*Proof:* According to the error evolution (5), we can find each element of the matrix  $D(t)$  has the same dynamics, then we have

$$\dot{D}_{iw}(t) = q\Psi(D_{iw}(t)), \quad (12)$$



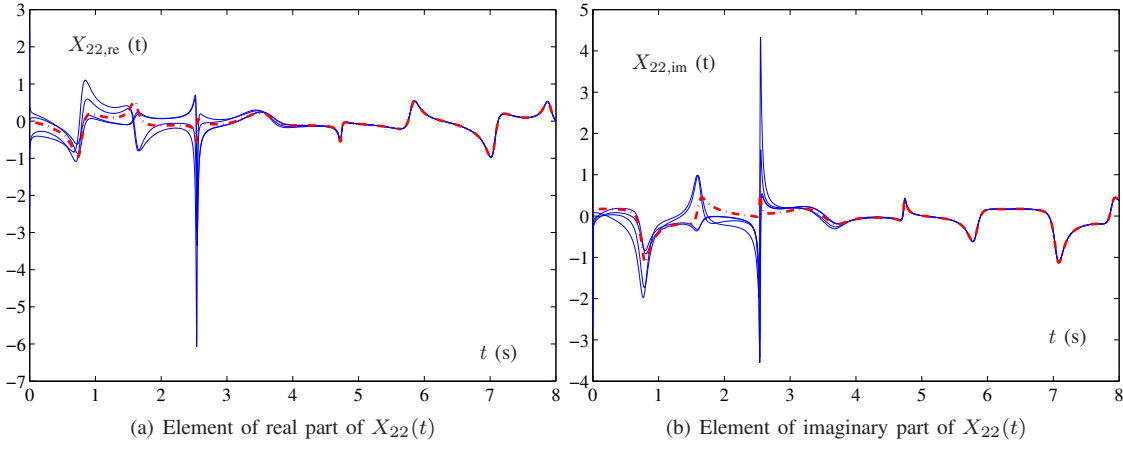


Fig. 9. Output trajectories of neural states  $X_{22}(t)$  synthesized by the model (10) in example 1. The dotted red line denotes the theoretical values, and the blue solid line denotes the calculated values.

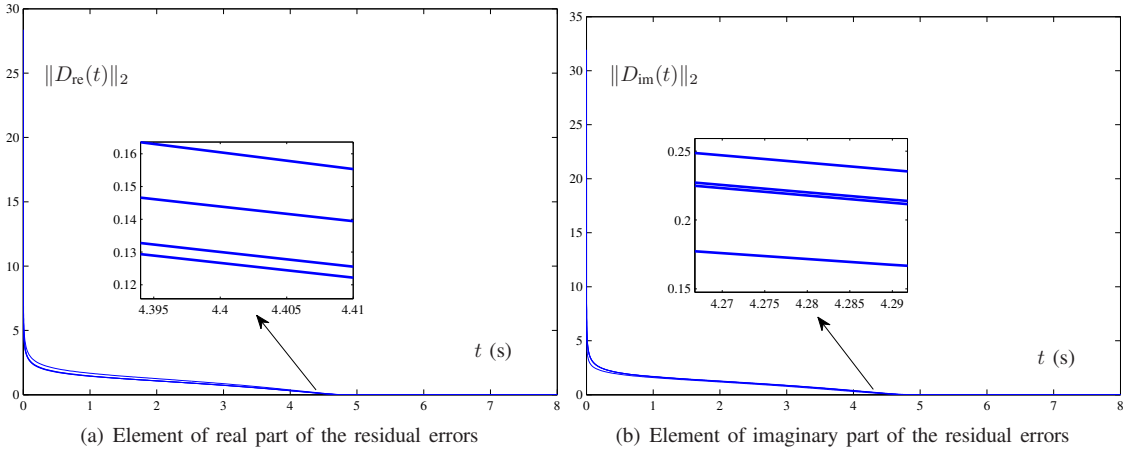


Fig. 10. Output trajectories of the residual errors synthesized by the model (10) in example 1.

where  $D_{iw}(t)$  denotes the  $iw$ th element of the error matrix  $D(t)$ . According to  $D_{iw}(t) = D_{iw, \text{re}}(t) + jD_{iw, \text{im}}(t)$ , the following two real-valued equations are derived:

$$\begin{aligned} \dot{D}_{iw, \text{re}}(t) &= q\Psi(D_{iw, \text{re}}(t)), \\ \dot{D}_{iw, \text{im}}(t) &= q\Psi(D_{iw, \text{im}}(t)), \end{aligned} \quad (13)$$

where  $D_{iw, \text{re}}(t)$  and  $D_{iw, \text{im}}(t)$  denote the real part and imaginary part of  $D_{iw}(t)$ , respectively. Then we can design the following Lyapunov functions:

$$\begin{aligned} V_{\text{re}}(t) &= D_{iw, \text{re}}^2(t), \\ V_{\text{im}}(t) &= D_{iw, \text{im}}^2(t). \end{aligned} \quad (14)$$

Considering the equation  $V_{\text{re}}(t) = D_{iw, \text{re}}^2(t)$  and the equation  $V_{\text{im}}(t) = D_{iw, \text{im}}^2(t)$  have the identical dynamic, we need only take the equation  $V_{\text{re}}(t) = D_{iw, \text{re}}^2(t)$  as a example to analyse the convergence property. Now we have

$$\dot{V}_{\text{re}}(t) = -2qD_{iw, \text{re}}(t)\Psi(D_{iw, \text{re}}(t)). \quad (15)$$

If we choose the sign-multi-power activation function, we will have

$$\begin{aligned} \Psi(D_{iw, \text{re}}(t)) &= a_1 \text{sgn}^z(D_{iw, \text{re}}(t)) \\ &+ a_2 \text{sgn}^{z-2^1}(D_{iw, \text{re}}(t)) \\ &+ a_3 \text{sgn}^{z-2^2}(D_{iw, \text{re}}(t)) \\ &+ \dots + a_n \text{sgn}^1(D_{iw, \text{re}}(t)) \\ &+ a_{n+1} \text{sgn}^{1/z}(D_{iw, \text{re}}(t)). \end{aligned} \quad (16)$$

From the equation (16), we can first find  $\text{sgn}^z(t)$ ,  $\dots$  and  $\text{sgn}^{1/z}(t)$  are monotone increasing functions. Then the equation (16) is an odd and monotone increasing function. So  $D_{iw, \text{re}}(t)\Psi(D_{iw, \text{re}}(t))$  is positive definite. Now according to the equation (15),  $\dot{V}$  is negative definite. Then the corresponding conclusion can be given that  $D_{iw, \text{re}}(t)$  will converge to 0 globally with time for all  $i$  and  $w$ . Similarly, we can prove the convergence property of  $D_{iw, \text{im}}(t)$  for all  $i$  and  $w$ .

Now from the equation (4), the corresponding conclusions can be given that the  $X(t)$  of the sign-multi-power model (11) will also converge to 0 globally. This proof is successful. ■

**Theorem 2:** The state  $X(t)$  of ICZNN model (11) will obtain its theoretical solution within the time  $t_h$ :

$$t_h = \frac{z}{q(z-1)} V_{\text{max}}^{\frac{z-1}{2z}}(0)$$

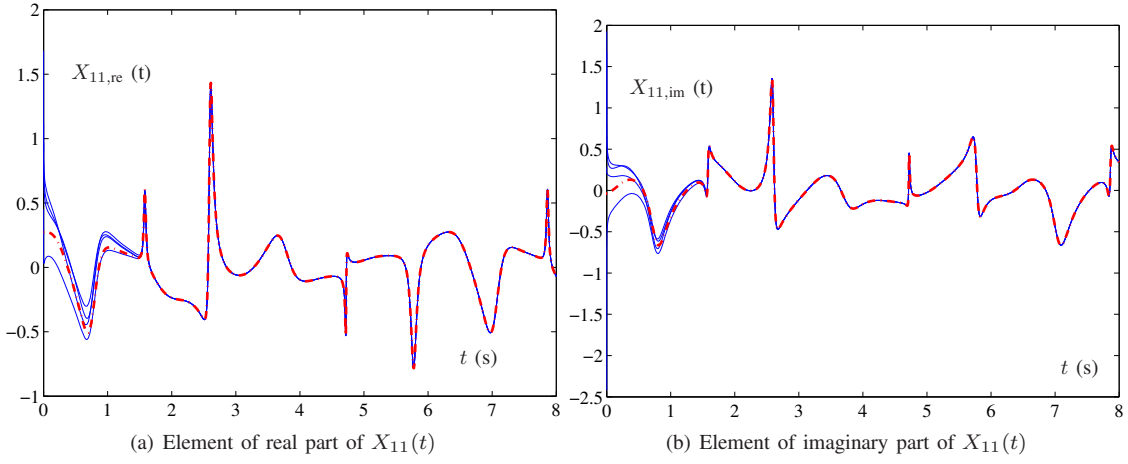


Fig. 11. Output trajectories of neural states  $X_{11}(t)$  synthesized by the model (11) in example 1. The dotted red line denotes the theoretical values, and the blue solid line denotes the calculated values.

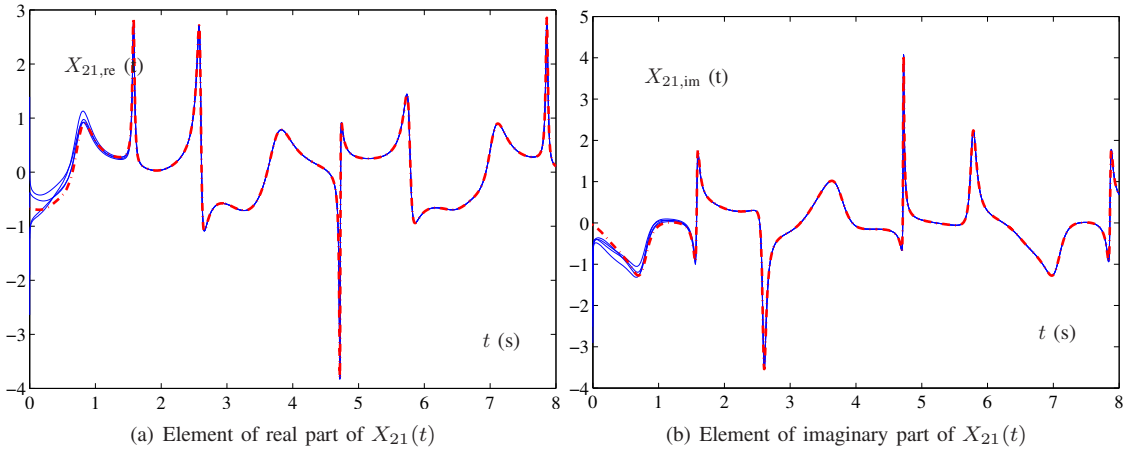


Fig. 12. Output trajectories of neural states  $X_{21}(t)$  synthesized by the model (11) in example 1. The dotted red line denotes the theoretical values, and the blue solid line denotes the calculated values.

where  $V_{max}(0)$  denotes the maximum initial element of  $D_{iw, re}(t)$  and  $D_{iw, im}(t)$  for all possible  $i$  and  $w$ , and  $z$  is an odd integer and satisfy  $z > 1$ .

*Proof:* According to the equation (13), we first design the following Lyapunov function:

$$\begin{aligned} V_{re}(t) &= D_{iw, re}^2(t), \\ V_{im}(t) &= D_{iw, im}^2(t). \end{aligned} \quad (17)$$

Now we take the equation  $V_{re}(t) = D_{iw, re}^2(t)$  as a example to

analyse the maximum convergence time, and have

$$\begin{aligned} \dot{V}_{re}(t) &= -2qD_{iw, re}(t)(a_1 \text{sgn}^z(D_{iw, re}(t))) \\ &\quad + a_2 \text{sgn}^{z-2^1}(D_{iw, re}(t)) + a_3 \text{sgn}^{z-2^2}(D_{iw, re}(t)) \\ &\quad + \dots + a_n \text{sgn}^1(D_{iw, re}(t)) \\ &\quad + a_{n+1} \text{sgn}^{1/z}(D_{iw, re}(t)) \\ &= -2q|D_{iw, re}(t)|(a_1 \text{sgn}^z(|D_{iw, re}(t)|)) \\ &\quad + a_2 \text{sgn}^{z-2^1}(|D_{iw, re}(t)|) \\ &\quad + a_3 \text{sgn}^{z-2^2}(|D_{iw, re}(t)|) \\ &\quad + \dots + a_n \text{sgn}^1(|D_{iw, re}(t)|) \\ &\quad + a_{n+1} \text{sgn}^{1/z}(|D_{iw, re}(t)|) \\ &\leq -2q|D_{iw, re}(t)|(a_{n+1} \text{sgn}^{1/z}(|D_{iw, re}(t)|)) \\ &= -2qV_{re}^{1/2}(t)(a_{n+1} \text{sgn}^{1/z}(V_{re}^{1/2}(t))). \end{aligned} \quad (18)$$

Then we have

$$\begin{aligned} \dot{V}_{re}(t) &\leq -2qV_{re}^{1/2}(t)a_{n+1} \text{sgn}^{1/z}(V_{re}^{1/2}(t)) \\ &= -2qV_{re}^{1/2}(t)V_{re}^{1/2z}(t) \\ &= -2qV_{re}^{\frac{1+z}{2z}}(t). \end{aligned} \quad (19)$$

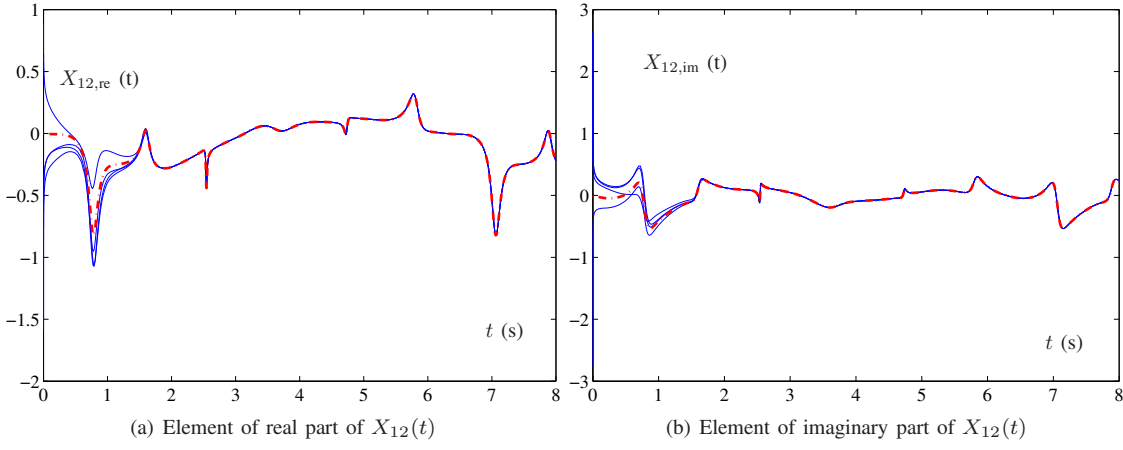


Fig. 13. Output trajectories of neural states  $X_{12}(t)$  synthesized by the model (11) in example 1. The dotted red line denotes the theoretical values, and the blue solid line denotes the calculated values.

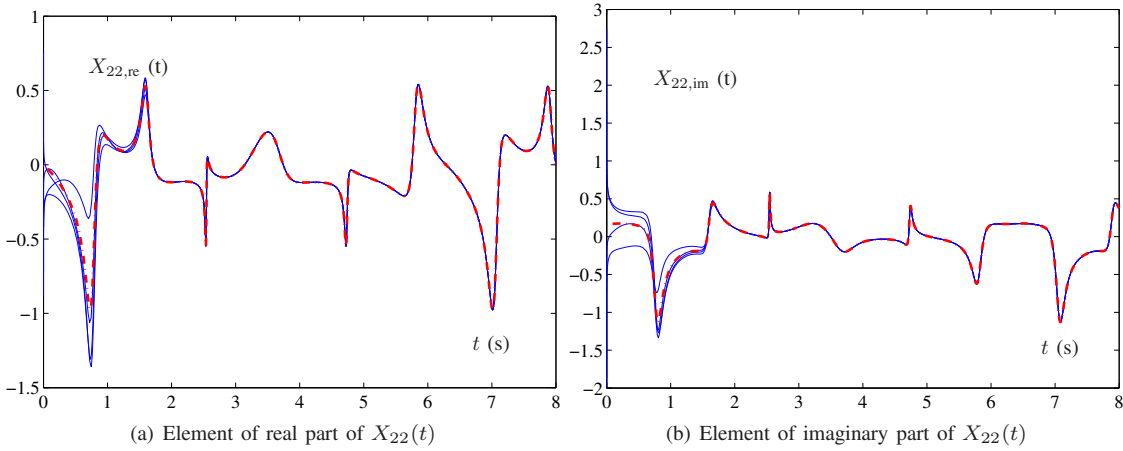


Fig. 14. Output trajectories of neural states  $X_{22}(t)$  synthesized by the model (11) in example 1. The dotted red line denotes the theoretical values, and the blue solid line denotes the calculated values.

Now let's integrate the equation (19) from zero to  $t$ , and we will have

$$V_{\text{re}}(t) \leq (V_{\text{re}}^{\frac{z-1}{2z}}(0) - \frac{q(z-1)}{z} t)^{\frac{2z}{z-1}}.$$

Let

$$t_{1,i} = \frac{z}{q(z-1)} v^{\frac{z-1}{2z}}(0). \quad (20)$$

Now we can draw a conclusion if  $t \geq t_{1,i}$ ,  $V_{\text{re}}(t) = 0$ . Suppose  $V_{\text{re},\text{max}}(0)$  denote the maximum element of  $D_{iw,\text{re}}(t)$  for all possible  $i$  and  $w$ , and  $t_{1,\text{re}} = \frac{z}{q(z-1)} V_{\text{re},\text{max}}^{\frac{z-1}{2z}}(0)$ . Therefore if  $t \geq t_{1,\text{re}}$ ,  $V_{\text{re}}(t) = 0$ . Similarly, we can deal with the Lyapunov function  $V_{\text{im}}(t) = D_{iw,\text{im}}^2(t)$  using the above method. Suppose  $V_{\text{im},\text{max}}(0)$  denote the maximum element of  $D_{iw,\text{im}}(t)$  for all possible  $i$  and  $w$ , and  $t_{1,\text{im}} = \frac{z}{q(z-1)} V_{\text{im},\text{max}}^{\frac{z-1}{2z}}(0)$ . Then we can find if  $t \geq t_{1,\text{im}}$ ,  $V_{\text{im}}(t) = 0$ .

Suppose  $t_1 = \max(t_{1,\text{re}}, t_{1,\text{im}})$ , and we can draw a conclusion that if  $t \geq t_1$ , the Lyapunov function (17) will converge to zero.

Now the proof is successful.  $\blacksquare$

#### IV. NUMERICAL SIMULATION

Now, two illustrative examples are provided in this section. Furthermore to show the advantage of the ICZNN model (11), the sign-bi-power model (9) and the IZNN model (10) are also used to calculate the solution of CVTVSE. The convergence process of each neural-state solution and the residual error norm of each model are shown in corresponding figures. For the convenience of comparison, the following parameters are chosen  $q = 1$ ,  $z = h = 5$ ,  $a_1 = a_2 = a_3 = a_4 = j_1 = j_2 = \frac{1}{2}$ , and  $j_3 = 0.25$ .

Example 1: In this example, the sign-bi-power model (9), the IZNN model (10) and the ICZNN model (11) are employed to calculate the theoretical solution  $\tilde{X}(t)$ , respectively. The specific CVTVSE example is presented as below:

$$G_1(t)X(t) - X(t)Q_1(t) = -S_1(t) \in \mathbb{C}^{n \times n},$$

where

$$G_1 = \begin{bmatrix} \cos(5t) + j4 \sin(2t) & 3 \sin(4t) + j6 \cos(3t) \\ 6 - \sin(t) + j \cos(4t) & 2 + \cos(2t) + j3 \sin(2t) \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

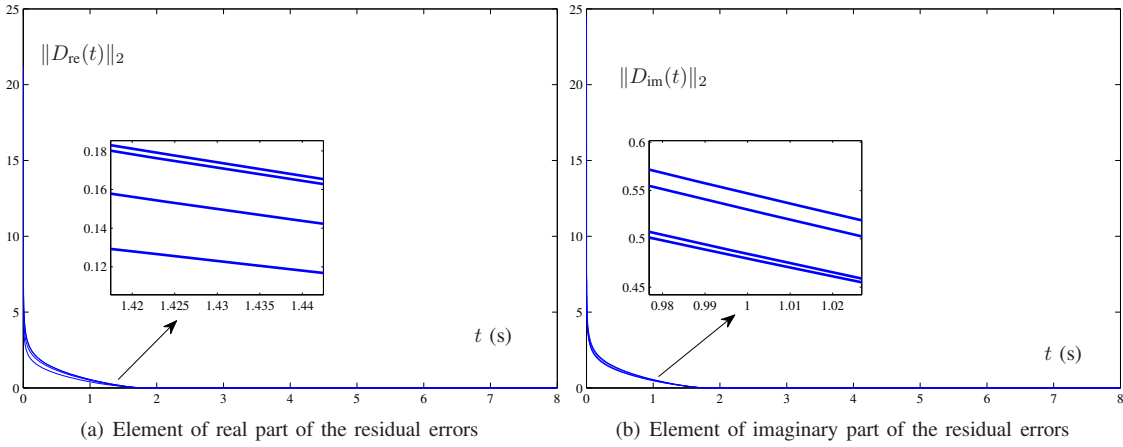


Fig. 15. Output trajectories of the residual errors synthesized by the model (11) in example 1.

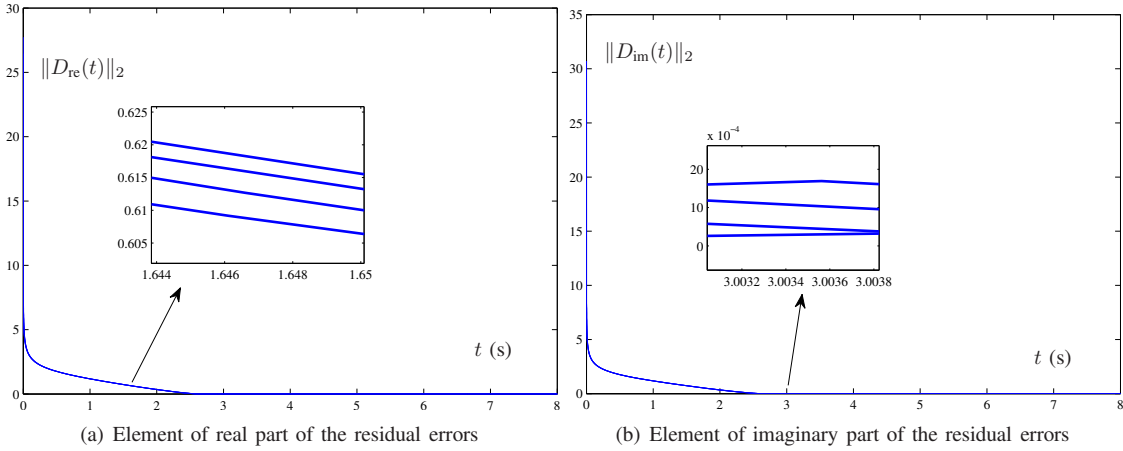


Fig. 16. Output trajectories of the residual errors synthesized by the model (9) in example 2.

and

$$S_1 = \begin{bmatrix} \sin(3t) & \sin(2t) \\ -\cos(t) + j4\sin(3t) & \sin(t) \end{bmatrix}.$$

The calculation results are displayed in Figs.1-15. The Figs. 1-4, the Figs. 6-9 and the Figs. 11-14 display the output trajectories of neural state  $X(t)$ . From these figures, we can find that the complex-valued time-varying parameters  $X_{11}(t)$ ,  $X_{21}(t)$ ,  $X_{12}(t)$ ,  $X_{22}(t)$  of different models will all converge to the theoretical solution in finite time. However, the ICZNN model (11) has the highest convergence speed. The Fig. 5, the Fig. 10 and the Figs. 15-18 display the output trajectories of the residual error norm  $\|D(t)\|_2$ . From these figures, we can find that the convergence time of the sign-bi-power model (9), the IZNN model (10), and the ICZNN model (11) is about 2.6s, 4.7s, and 1.7s, respectively. Compared with the sign-bi-power model (9) and the IZNN model (10), the ICZNN model increases the convergence speed about percent 34% and 62%, respectively.

Example 2: In this example, the sign-bi-power model (9), the IZNN model (10) and the sign-multi-power model (11) are further employed to calculate the following CTVTSE:

$$G_2(t)X(t) - X(t)Q_2(t) = -S_2(t) \in \mathbb{C}^{n \times n},$$

where

$$G_2 = \begin{bmatrix} \cos(t) + j3\sin(4t) & 5\sin(3t) + j\cos(4t) \\ 8 - j\cos(4t) & 6\cos(3t) + j\sin(4t) \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 2 + j\cos(2t) & 4 + j\sin(3t) \\ \cos(3t) - j\sin(2t) & 3 + \cos(t) \end{bmatrix},$$

and

$$S_2 = \begin{bmatrix} \sin(t) + j4\cos(t) & \cos(t) \\ -\cos(t) + j\sin(3t) & \sin(t) + j\sin(2t) \end{bmatrix}.$$

The calculation results are displayed in Fig.16-18 which show the output trajectories of the residual error norm  $\|D(t)\|_2$ . Similarly, we can find the the sign-multi-power model (11) has a higher convergence speed than the sign-bi-power model (9) and the IZNN model (10).

## V. CONCLUSIONS

In this paper a novel activation function is designed. Based on this novel activation function, a new finite-time ZNN model (11) is designed to tackle the CTVTSE, and the strict proof of global convergence and the upper bound are described. The simulation results validate the proposed ICZNN model can

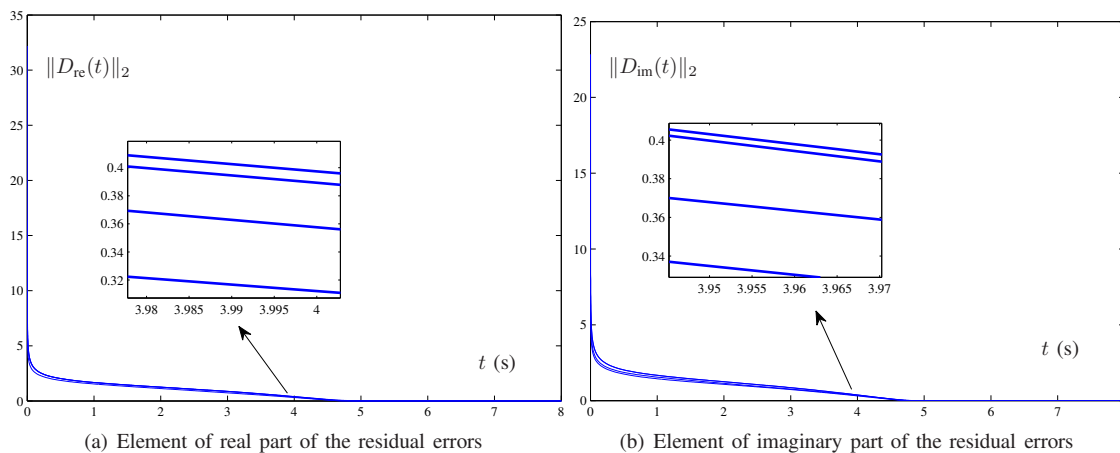


Fig. 17. Output trajectories of the residual errors synthesized by the model (10) in example 2.

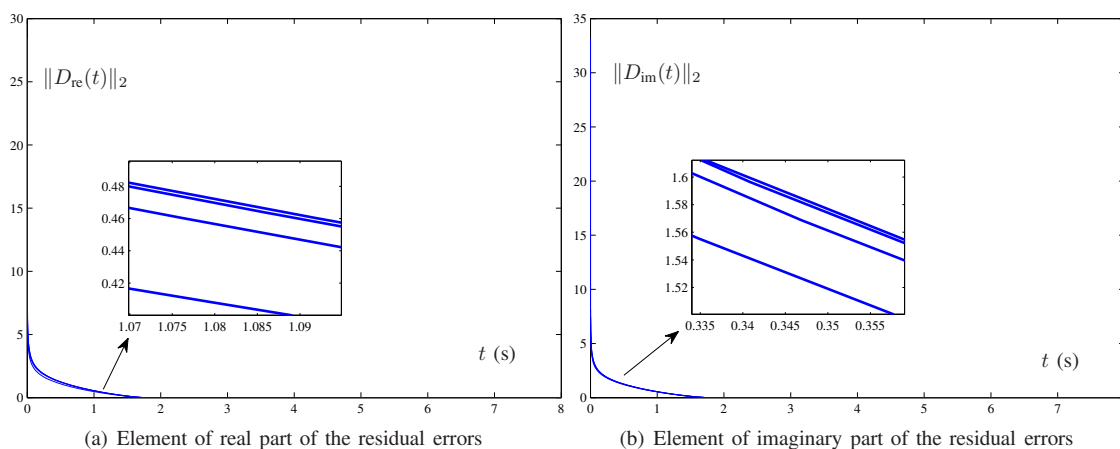


Fig. 18. Output trajectories of the residual errors synthesized by the model (11) in example 2.

increase the convergence speed significantly. So it has a certain significance for tackling the CVTVSE online.

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