



On the number of equilibria of the replicator-mutator dynamics for noisy social dilemmas

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ABSTRACT

In this paper, we consider the replicator-mutator dynamics for pairwise social dilemmas where the payoff entries are random variables. The randomness is incorporated to take into account the uncertainty, which is inevitable in practical applications and may arise from different sources such as lack of data for measuring the outcomes, noisy and rapidly changing environments, as well as unavoidable human estimate errors. We analytically and numerically compute the probability that the replicator-mutator dynamics has a given number of equilibria for four classes of pairwise social dilemmas (Prisoner's Dilemma, Snow-Drift Game, Stag-Hunt Game and Harmony Game). As a result, we characterise the qualitative behaviour of such probabilities as a function of the mutation rate. Our results clearly show the influence of the mutation rate and the uncertainty in the payoff matrix definition on the number of equilibria in these games. Overall, our analysis has provided novel theoretical contributions to the understanding of the impact of uncertainty on the behavioural diversity in a complex dynamical system.

1. Introduction

Evolutionary game theory (EGT), which combines the analysis of game theory with that of dynamic evolutionary processes, provides a powerful mathematical and simulation framework for the study of dynamics of frequencies of competing strategies in large populations [1–3]. This framework has been successfully used for the investigation of the evolution of collective behaviours such as cooperation, coordination, trust and fairness, and recently, for understanding several pressing societal challenges such as climate change and pandemics mitigation and advanced technology governance [4–10].

However, existing works on evolutionary game theory mainly focus on deterministic games which could not capture the different random factors that define the interactions, in particular the game payoff matrix [4,5,11]. That is, these works assume that the outcomes of interactions for any group of strategists (aka the game payoff matrix) can be defined in advance with certainty. However, the uncertainty in defining such outcomes is in general unavoidable which can arise from a diversity of possible sources, including lack of data for measuring the outcomes, noisy and rapidly changing environments, as well as unavoidable human estimate errors [12–15].

On the other hand, existing works that capture such uncertainty factors in EGT usually assume the payoff entries of the game are general random variables whose distributions are not known a priori [16–20]. In this line, random social dilemma games where the payoff entries satisfy certain ordering required for particular social dilemmas were also analysed [20,21]. These approaches are useful to provide generic properties of the underlying dynamical systems, including patterns of evolutionary stable equilibria [22] and the statistical properties of the quantity and distribution of the (stable) equilibrium points [16–18,23]. Additionally, random games were used to study which population structure can strongly promote cooperation [24].

However, it might be the case that in many domains/scenarios, some knowledge about the payoff entries is available. In particular, some of the payoff entries might fluctuate around certain known values, which are for example estimated through data analysis or given by domain experts. Capturing the available information in the analysis is essential to more accurately describe system dynamics and evolutionary outcomes. For example, several experimental studies estimate the entries in the fitness matrix from in vitro data of interactions between microbial populations, for evolutionary game theory analysis [25–27].

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In this paper, we bridge this gap by analytically investigating the statistic of the number of equilibrium points in pairwise social dilemma games where some payoff entries are drawn from a random distribution with a known mean value and variance. We will consider the full space of (symmetric) two-player games in which players can choose either to cooperate or defect (see detailed definitions in Section 1.2) [4,28]. These games have been shown to provide important abstract frameworks to capture collective behaviours in a wide range of biological and social interactions such as cooperation and (anti-)coordination. We adopt in our analysis the replicator-mutator dynamics (see Sections 2.1 and 2.2) to model the evolutionary process, capturing both selection and mutation, allowing us to generate insights on how uncertainty in the interaction outcomes and these stochastic factors together influence equilibrium outcomes. Note that most previous works on random games consider replicator equations [16–18], which is a simple version of the replicator-mutator ones where mutation is assumed to be negligible, already general enough to encompass a variety of biological contexts from ecology to population genetics and from prebiotic to social evolution [29]. A similar setting to our work was considered in [30,31] for studying the impact of uncertainty on the evolution of cooperation, but their analysis was purely based on simulations and did not use replicator or replicator-mutator equations. Moreover, there has been a particular interest in studying average and maximal numbers of equilibrium points in a dynamical system [18,32,33]. Our analysis provides close forms for the probability of a concrete number of equilibria to occur, thus generalising these results.

The rest of the paper is organised as follows. In Section 2 we provide the background regarding replicator-mutator dynamics and summarise the main results of the paper. Detailed proofs will follow in Sections 3–5.

2. Background and main results

2.1. The replicator-mutator dynamics

The replicator-mutator equation describes the evolution dynamics in a population of different strategies being in co-presence, where selection and mutation are both captured. It is an well established mathematical framework that integrates the unavoidable mutation observed in various biological and social settings [34–39]. This framework has been utilised in many application domains, including evolution of collective behaviours [11,40], social networks dynamics [41], language evolution [42], population genetics [43], and autocatalytic reaction networks [44].

We consider an infinitely large population consisting of n different strategies S_1, \dots, S_n . Their frequencies are denoted, respectively, by x_1, \dots, x_n , where $\sum_{i=1}^n x_i = 1$. These strategies undergo selection where their frequency, S_i , is determined by its fitness (i.e. average payoff), f_i , which is obtained through interactions with others in the population. Such interactions happen within randomly selected pairs of individuals playing a social dilemma game (see details below). By means of mutation, individuals in the population might change their strategy to another randomly selected strategy, given by the so-called mutation matrix: $Q = (q_{ji}), j, i \in \{1, \dots, n\}$. Here, q_{ji} stands for the probability of an S_j individual changing its strategy to S_i , satisfying that

$$\sum_{j=1}^n q_{ji} = 1, \quad \forall i \leq n. \tag{1}$$

Denoting vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\bar{f}(\mathbf{x}) = \sum_{i=1}^n x_i f_i(\mathbf{x})$ the population's average fitness, we can describe the replicator-mutator equation as follows [45–48]

$$\dot{x}_i = \sum_{j=1}^n x_j f_j(\mathbf{x}) q_{ji} - x_i \bar{f}(\mathbf{x}), \quad 1 \leq i \leq n. \tag{2}$$

It is important to note that the replicator dynamics can be reproduced from (2) with Q being the identity matrix (i.e. no mutation). This paper

investigates the equilibrium points of the replicator-mutator dynamics, which are solutions in $[0, 1]^n$ (more concretely, on a $(n-1)$ -dimensional simplex satisfying the normalisation $\sum_{i=1}^n x_i = 1$) of the following system of equations

$$g_i(\mathbf{x}) = 0, \quad 1 \leq i \leq n,$$

where $g_i(\mathbf{x}), i = 1, \dots, n$, denotes the right-hand side of (2)

$$g_i(\mathbf{x}) := \sum_{j=1}^n x_j f_j(\mathbf{x}) q_{ji} - x_i \bar{f}(\mathbf{x}).$$

In general, knowing equilibrium points in a dynamical system allows us to study states where different strategies might co-exist in the population, indicating the possibility of polymorphism. However, for $n > 2$ this is a challenging problem since one needs to deal with a system of multivariate nonlinear equations. In this paper, we will focus on two-player two-strategy games, but with random payoff entries, and characterise the statistics of the number of equilibrium points.

2.2. Pairwise social dilemmas

Now let us consider a pairwise game with two strategies S_1 and S_2 with a general payoff matrix and mutation matrix given below

$$S_1 \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad Q = \begin{matrix} S_1 & S_2 \\ S_1 & S_2 \end{matrix} \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$$

where $a_{11} \in \mathbb{R}$ is the payoff that a player using strategy S_1 obtains when interacting with another player, who is also using strategy S_1 . Other notations are interpreted similarly. Denoting $x, x \in [0, 1]$, as S_1 's frequency (and thus $1 - x$ as S_2 's frequency), the fitness of the strategy S_1 and S_2 , as well as of the population, are given by

$$f_1(x) = a_{11}x + a_{12}(1 - x), \quad f_2(x) = a_{21}x + a_{22}(1 - x),$$

and $\bar{f}(x) = x f_1(x) + (1 - x) f_2(x)$.

Substituting these expressions into (2), we obtain the following replicator-mutator equation for a general two-player two-strategy game

$$\dot{x} = q_{11} a_{11} x^2 + q_{11} x(1 - x) a_{12} + q_{21} x(1 - x) a_{21} + q_{21} a_{22} (1 - x)^2 - x (a_{11} x^2 + (a_{12} + a_{21}) x(1 - x) + a_{22} (1 - x)^2). \tag{3}$$

We assume that the mutation matrix is symmetric, that is $q_{12} = q_{21} =: q$, where $0 \leq q \leq 0.5$ models the strength of the mutation. Thus, according to (1), $q_{11} = q_{22} = 1 - q$. Eq. (3) is reduced to

$$\dot{x} = (a_{12} + a_{21} - a_{11} - a_{22}) x^3 + (a_{11} - a_{21} - 2(a_{12} - a_{22}) + q(a_{22} + a_{12} - a_{11} - a_{21})) x^2 + (a_{12} - a_{22} + q(a_{21} - a_{12} - 2a_{22})) x + qa_{22}. \tag{4}$$

It can be seen that the right hand side of Eq. (4) is reduced to qa_{22} when $x = 0$ and to $-qa_{11}$ when $x = 1$. Thus, for non-zero mutation (i.e. $q > 0$), $x = 0$ and $x = 1$ are not equilibrium points when a_{11} or a_{22} is not equal 0, respectively. It is also noteworthy that only the differences $a_{21} - a_{11}$ and $a_{12} - a_{22}$ matter for determining the equilibria when there is no mutation (i.e. $q = 0$).

Next, we consider a well-established parameterisation of pairwise social dilemmas, for random game analysis [49–51]. In these games, players can choose to cooperate or defect in each interaction. Mutual cooperation (punishment) would lead to a payoff R (P). Unilateral cooperation leads to payoff S while unilateral defection leads to payoff T . Without loss of generality, we normalise $R = 1$ and $P = 0$ in all games, and that $0 \leq a_{21} = T \leq 2$ and $-1 \leq a_{12} = S \leq 1$. We focus on four important social dilemma games, characterised by different orderings of the payoff entries

- (i) the Prisoner’s Dilemma (PD): $2 \geq T > 1 > 0 > S \geq -1$ (both players defect),
- (ii) the Snow-Drift (SD) game: $2 \geq T > 1 > S > 0$ (players prefer unilateral defection to mutual cooperation),
- (iii) the Stag Hunt (SH) game: $1 > T > 0 > S \geq -1$ (players prefer mutual defection to unilateral cooperation),
- (iv) the Harmony (H) game: $1 > T \geq 0, 1 \geq S > 0$ (both players cooperate).

As motivated in the introduction, to be more realistic and to capture the various possible uncertainty, we will consider random games, where T or S or both T and S are random variables. Hence, the number of equilibrium points for these social dilemma games is also a random variable. The aim of this paper is to compute its probability distribution, as a function of the mutation strength q .

2.3. Main results

Our main results, Theorems 1 and 2 below, provide explicit formulas for the probability distribution, that is the probability that the replicator-mutator dynamics has a certain number of equilibria, for the four social dilemmas above where T or S is random and the other is fixed. We also numerically investigate the case where both T and S are random. The distinction between these cases is necessary since T and S play different roles in the games, due to the fact that R is selected to be higher than P and this has an effect on how mutation affects payoffs and equilibria. The obtained explicit formulas also enable us to derive interesting qualitative behaviours of the probabilities. Overall, our results clearly show the non-trivial influence of the mutation strength and the role of T and S in the equilibrium outcomes.

We now summarise the main analytical results of the paper. See subsequent sections for further details and other qualitative results.

Theorem 1 (T is Random). *Suppose that T is normally distributed with mean T_0 and variance σ^2 , i.e. $T \sim \mathcal{N}(T_0, \sigma^2)$, and S is a given number where T_0 and S satisfy the corresponding ordering in each of the social dilemmas above. Let T_1, T_2, T_3 and s_1, s_2 be defined in (11), (12) and (13). Then, for all games, the probability that the replicator-mutator has two equilibria is given by*

$$p_2 = \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{T_3 - T_0}{\sigma\sqrt{2}}\right).$$

The probability that the replicator-mutator has three equilibria is given as follows. First, for SD and H games

$$p_3 = 1 - p_2.$$

Now, for PD and SH games

$$p_3 = \begin{cases} \frac{1}{2} \left[1 + \operatorname{erf}\left(\frac{T_1 - T_0}{\sigma\sqrt{2}}\right) \right], & \text{if } S \leq s_1, \\ \frac{1}{2} \left[1 + \operatorname{erf}\left(\frac{T_1 - T_0}{\sigma\sqrt{2}}\right) + \operatorname{erf}\left(\frac{T_3 - T_0}{\sigma\sqrt{2}}\right) \right] \\ - \operatorname{erf}\left(\frac{T_2 - T_0}{\sigma\sqrt{2}}\right), & \text{if } s_1 < S < s_2, \\ \frac{1}{2} \left[1 + \operatorname{erf}\left(\frac{T_3 - T_0}{\sigma\sqrt{2}}\right) \right], & \text{if } S \geq s_2. \end{cases}$$

Thus the probability that the replicator-mutator has one equilibrium is, for SD and H games $p_1 = 0$, and for PD and SH games, $p_1 = 1 - p_2 - p_3$.

Furthermore, as a function of q , the probability p_2 is decreasing in SD and H games, but is increasing in PD and SH games and satisfies the following small mutation limit

$$\lim_{q \rightarrow 0} p_2 = \begin{cases} 1, & \text{in SD and H games,} \\ 0, & \text{in PD and SH games.} \end{cases}$$

In addition, in SD and H games, it holds that

$$p_2 \geq \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(-\frac{S + T_0}{\sigma\sqrt{2}}\right) > \frac{1}{2}.$$

Theorem 2 (S is Random). *Suppose that S is normally distributed with mean S_0 and variance σ^2 , i.e. $S \sim \mathcal{N}(S_0, \sigma^2)$, and T_0 is a given number so that T_0 and S satisfy the corresponding ordering in each of the social dilemmas above. Let S_1, S_2 be defined in (20). The probabilities that the replicator-mutator dynamics has 1, 2 and 3 equilibria are given by, respectively,*

$$p_1 = 1 - p_2 - p_3,$$

$$p_2 = \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(-\frac{\frac{qT_0}{1-q} - S_0}{\sqrt{2}\eta}\right),$$

$$p_3 = \begin{cases} \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{S_1 - S_0}{\sqrt{2}\eta}\right), & \text{if } T_0 > \frac{(1-q)^2}{1-2q}, \\ \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{S_1 - S_0}{\sqrt{2}\eta}\right) + \frac{1}{2} \operatorname{erf}\left(\frac{-\frac{qT_0}{1-q} - S_0}{\sqrt{2}}\eta\right) \\ - \frac{1}{2} \operatorname{erf}\left(\frac{S_2 - S_0}{\sqrt{2}\eta}\right), & \text{if } T_0 < \frac{(1-q)^2}{1-2q}. \end{cases}$$

As a consequence, p_2 is always increasing as a function of q and satisfies the following lower and upper bounds

$$\frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{-S_0}{\sqrt{2}\eta}\right) < p_2 < \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{-T_0 - S_0}{\sqrt{2}\eta}\right).$$

In the following we will provide detailed proofs for these main results. In Section 3 we study the case where T is random and S is deterministic. We compute the probability that the replicator-mutator dynamics has p_k ($k \in \{1, 2, 3\}$) equilibria, both analytically and numerically by sampling the payoff matrix space. Similar results where S is random and T is deterministic are obtained in Section 4. In Section 5 we numerically investigate the case where both T and S are random. A summary and outlook is given in the final section, Section 6.

3. T is random

We first consider the case where only T is random. Specifically, we assume that

$$T = T_0 + \varepsilon_T, \tag{5}$$

where ε_T is a centred random variable. Suppose that ε_T is a centred normal distribution, $\varepsilon_T \sim \mathcal{N}(0, \sigma^2)$, and T_0 is a fixed number. It follows that

$$T \sim \mathcal{N}(T_0, \sigma^2).$$

This means that we have partial information about the value of T , which is randomly fluctuating (perturbed) around a deterministic value T_0 . In practical applications, this may come from estimations based on an expert’s advice or data simulations. By taking the value of the variance smaller and smaller, the value of T is more and more concentrated around T_0 , and in particular, by sending the variance to zero, we recover deterministic games.

Note that, the choice of the fixed value of S will partially determine which game is played. Indeed, when $S < 0$ the game will either be a PD or SH, and when $S > 0$, it will either be an H or HD game. We anticipate that the qualitative outcomes will exhibit a significant dependence on the sign of S .

3.1. Equilibrium points

By simplifying the right hand side of (4), equilibria of a social dilemma game are roots in the interval $[0, 1]$ of the following cubic

equation

$$(T + S - 1)x^3 + (1 - T - 2S + q(S - 1 - T))x^2 + (S + q(T - S))x = 0. \quad (6)$$

It follows that $x = 0$ is always an equilibrium.

If $q = 0$, (6) reduces to

$$(T + S - 1)x^3 + (1 - T - 2S)x^2 + Sx = 0,$$

which has solutions

$$x = 0, \quad x = 1, \quad x^* = \frac{S}{S + T - 1}.$$

Note that for SH and SD games $x^* \in (0, 1)$, thus it is always an (internal) equilibrium. On the other hand, for PD-games and H-games, $x^* \notin (0, 1)$, thus it is not an equilibrium.

If $q = \frac{1}{2}$ then the above equation has two solutions $x_1 = \frac{1}{2}$ and $x_2 = \frac{T+S}{T+S-1}$. In PD, SD and H games, $x_2 \notin (0, 1)$, thus they have two equilibria $x_0 = 0$ and $x_1 = \frac{1}{2}$. In the SH game: if $T + S < 0$ then the game has three equilibria $x_0 = 0, x_1 = \frac{1}{2}$ and $0 < x_2 < 1$; if $T + S \geq 0$ then the game has only two equilibria $x_0 = 0, x_1 = \frac{1}{2}$.

Now we consider the case $0 < q < \frac{1}{2}$. For non-zero equilibrium points we solve the following quadratic equation

$$h(x) := (T+S-1)x^2 + (1-T-2S+q(S-1-T))x + S + q(T-S) =: ax^2 + bx + c = 0, \quad (7)$$

where we define

$$a = T + S - 1, \quad b = 1 - T - 2S + q(S - 1 - T), \quad c = S + q(T - S). \quad (8)$$

Note that for $q > 0$, $h(1) = -q < 0$. Thus, in contrast to the replicator dynamics [52], $x = 1$ is no longer an equilibrium in the replicator-mutator dynamics for all social dilemma games when $q > 0$. The fact that for $q > 0$, $x = 0$ is an equilibrium is due to the assumption $a_{22} = P = 0$, while $x = 1$ is not an equilibrium is due to the assumption $a_{11} = R > P = a_{22} = 0$, a property whereby mutual cooperation is more beneficial than mutual defection for social dilemma interactions. Therefore we only need to seek for internal equilibria $0 < x < 1$.

Set $t := \frac{x}{1-x}$, then we obtain

$$\frac{h(x)}{(1-x)^2} = (a + b + c)t^2 + (b + 2c)t + c = -qt^2 + (-q - a + c)t + c := g(t).$$

Thus, an equilibrium point of a social dilemma can be found from a positive solution of the following quadratic equation

$$g(t) = -qt^2 + (-q - a + c)t + c = -qt^2 - ((1-q)(T-1) + qS)t + S + q(T-S). \quad (9)$$

Using this relation, subsequently we provide numerical simulations and analytical results for the probability p_k that each of the game mentioned above has a certain number, $k \in \{1, 2, 3\}$, of equilibria. For the quadratic function $g(t)$, the discriminant is given by

$$\Delta = (q + a - c)^2 + 4qc, \quad (10)$$

where a and c are defined in terms of T and S in (8). The number of positive roots of g is characterised in the following three cases:

- (a) g has no positive roots, which happens when g has no real roots ($\Delta < 0$) or g has only negative roots ($\Delta \geq 0, t_1 \leq 0, t_2 \leq 0$). In this case, the replicator-mutator equation has only one equilibrium $x = 0$.
- (b) g has one positive root, which happens when g has a positive double root ($\Delta = 0, t_1 = t_2 > 0$) or when g has one positive and one negative root ($\Delta > 0, t_1 t_2 < 0$). In this case, the replicator-mutator equation has two equilibria.
- (c) g has two positive roots ($\Delta > 0, t_1 + t_2 > 0, t_1 t_2 > 0$), thus the replicator-mutator equation has three equilibria.

Using (8) we can write

$$\begin{aligned} \Delta &= (q + a - c)^2 + 4qc \\ &= [q + T + S - 1 - (S + qT - qS)]^2 + 4q[S + q(T - S)] \end{aligned}$$

$$\begin{aligned} &= (1 - q)^2 T^2 + 2[(q - q^2)S + q^2 + 2q - 1]T \\ &\quad + q^2 S^2 + 2q(1 - q)S + (q - 1)^2, \end{aligned}$$

which is a quadratic function of T whose discriminant is given by

$$2q^3 - q^2 - (2q^3 - 3q^2 + q)S = q(2q - 1)[q - (q - 1)S].$$

Therefore

- (i) If $2q^3 - q^2 - (2q^3 - 3q^2 + q)S > 0$, then $\Delta = 0$ has two distinct solutions.
- (ii) If $2q^3 - q^2 - (2q^3 - 3q^2 + q)S = 0$, then $\Delta = 0$ has only one solution.
- (iii) If $2q^3 - q^2 - (2q^3 - 3q^2 + q)S < 0$, then $\Delta = 0$ has no real solutions.

If $(q - 1)S \leq q$, then $\Delta = 0$ has solutions

$$T_1 = \frac{-((1 - q)qS + q^2 + 2q - 1) - 2\sqrt{2q^3 - q^2 - (2q^3 - 3q^2 + q)S}}{(1 - q)^2}, \quad (11)$$

$$T_2 = \frac{-((1 - q)qS + q^2 + 2q - 1) + 2\sqrt{2q^3 - q^2 - (2q^3 - 3q^2 + q)S}}{(1 - q)^2}. \quad (12)$$

In addition, we also have the following:

$$\begin{aligned} T_1 + T_2 &= \frac{q + a - c}{-q} = \frac{q + T + S - 1 - (S + qT - qS)}{-q} \\ &= \frac{q + T - qT + qS - 1}{-q}, \\ T_1 T_2 &= \frac{c}{-q} = \frac{S + qT - qS}{-q}. \end{aligned}$$

Define

$$s_1 = \frac{-q(1 - q)}{1 - 2q}, \quad s_2 = \frac{q}{q - 1}, \quad \text{and} \quad T_3 = \frac{(q - 1)S}{q}. \quad (13)$$

Note that since $0 < q < 1/2$, $s_1 \in (-\infty, 0)$, $s_2 \in (-1, 0)$ and $s_1 < s_2$. In addition, if $S = s_1$ then $T_2 = T_3$ and if $S = s_2$ then $T_1 = T_2$. Furthermore, the condition that $(q - 1)S \leq q$ will always hold when $S > 0$, that is for SD and H games.

3.2. Probability that the replicator-mutator has two equilibria

We first compute the probability p_2 that the replicator-mutator equation has two equilibria, which amounts to compute the probability that $\Delta = 0, t_1 = t_2 > 0$ or $\Delta > 0, t_1 t_2 < 0$:

$$p_2 = \mathbb{P}\{\Delta = 0, t_1 = t_2 > 0\} + \mathbb{P}\{\Delta > 0, t_1 t_2 < 0\} = p_{21} + p_{22}.$$

We note that when $\Delta = 0$ has no solution, $\mathbb{P}\{\Delta = 0, t_1 = t_2 > 0\} = 0$. When $\Delta = 0$ has two solutions T_1 and T_2 (T_1 may equal to T_2 here, but it does not matter), then $\mathbb{P}\{\Delta = 0, t_1 = t_2 > 0\}$ is given by

$$P(T = T_1) + P(T = T_2).$$

Since in both cases (Gaussian and uniform distributions) T follows a continuous distribution, we have $P(T = T_1) = P(T = T_2) = 0$. It follows that $p_{21} = 0$. Thus

$$p_2 = p_{22} = \mathbb{P}\{\Delta > 0, t_1 t_2 < 0\},$$

which is equivalent to the probability that $T > T_3$. Since $T \sim \mathcal{N}(T_0, \sigma^2)$, we obtain

$$\begin{aligned} p_2 &= \mathbb{P}(T > T_3) \\ &= \int_{T_3}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x - T_0)^2}{2\sigma^2}\right) dx \\ &= \frac{1}{\sqrt{\pi}} \int_{\frac{T_3 - T_0}{\sigma\sqrt{2}}}^{\infty} \exp(-t^2) dt \quad (\text{by changing of variable } t = \frac{x - T_0}{\sigma\sqrt{2}}) \\ &= \frac{1}{\sqrt{\pi}} \left[\int_0^{\infty} \exp(-t^2) dt - \int_0^{\frac{T_3 - T_0}{\sigma\sqrt{2}}} \exp(-t^2) dt \right] \\ &= \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{T_3 - T_0}{\sigma\sqrt{2}}\right) \end{aligned} \quad (14)$$

$$= \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{(q-1)S/q - T_0}{\sigma\sqrt{2}}\right),$$

where $\operatorname{erf}(\cdot)$ is the error function

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt.$$

The following lemma presents some interesting qualitative properties of p_2 .

Lemma 1. *As a function of q , the probability p_2 is decreasing in SD and H games, but is increasing in PD and SH games. As a consequence, for SD and H games, it holds that*

$$p_2 \geq \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(-\frac{S+T_0}{\sigma\sqrt{2}}\right) > \frac{1}{2}. \tag{15}$$

In addition,

$$\lim_{q \rightarrow 0} p_2 = \begin{cases} 1, & \text{in SD and H games,} \\ 0, & \text{in PD and SH games.} \end{cases}$$

It is worth mentioning that the monotonicity and the small mutation limits above are independent of the specific values of S, T_0 and σ . The lower bound (15) indicates that in SD and H games, p_2 is always dominant over p_1 and p_3 since $p_1 + p_2 + p_3 = 1$. For instance, in the SD game, with the specific values $S = 0.5, T_0 = 1.5, \sigma = 1$ we obtain

$$p_2 \geq \frac{1}{2} - \frac{1}{2} \operatorname{erf}(-\sqrt{2}) \approx 0.97725,$$

and in the H game with the specific values $S = 0.5, T = 0.5, \sigma = 1$, we obtain

$$p_2 \geq \frac{1}{2} - \frac{1}{2} \operatorname{erf}(-1/\sqrt{2}) \approx 0.8413.$$

Proof. Let $z = \frac{(q-1)S/q - T_0}{\sigma\sqrt{2}}$, then

$$p_2 = \frac{1}{2} - \frac{1}{2} \operatorname{erf}(z).$$

Therefore, using the chain rule and the fact that $\frac{d}{dz} \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} e^{-z^2}$, we obtain

$$\frac{dp_2}{dq} = \frac{dp_2}{dz} \frac{dz}{dq} = -\frac{1}{\sqrt{\pi}} e^{-z^2} \frac{S}{\sigma q^2 \sqrt{2}}.$$

Since S is positive in SD and H games but is negative in PD and SH games, $\frac{dp_2}{dq}$ is negative in SD and H games but is positive in PD and SH games. Thus the probability p_2 , as a function of q , is decreasing in SD and H games, but is increasing in PD and SH games. As a consequence, for SD and H games, we have

$$p_2 \geq p_2|_{q=1/2} = \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(-\frac{S+T_0}{\sigma\sqrt{2}}\right).$$

Now we establish the limit of p_2 as q tends to 0. Since $0 < q < \frac{1}{2}$ we have

$$\lim_{q \rightarrow 0} z = \begin{cases} -\infty, & \text{if } S > 0, \\ +\infty, & \text{if } S < 0. \end{cases}$$

Together with the fact that

$$\lim_{z \rightarrow \pm\infty} \operatorname{erf}(z) = \pm 1,$$

we obtain

$$\lim_{q \rightarrow 0} p_2 = \frac{1}{2} - \frac{1}{2} \lim_{q \rightarrow 0} \operatorname{erf}(z) = \begin{cases} 1, & \text{if } S > 0, \\ 0, & \text{if } S < 0. \end{cases}$$

Applying this to the underlying games, we achieve

$$\lim_{q \rightarrow 0} p_2 = \begin{cases} 1, & \text{in SD and H games,} \\ 0, & \text{in PD and SH games. } \square \end{cases}$$

3.3. Probability that the replicator-mutator has three equilibria

Now we compute the probability p_3 that the replicator-mutator equation has three equilibria. We consider the following cases, depending on the ordering between S and $s_1 < s_2$.

(C1) $S > q/(q-1) = s_2$

Then, recalling that when $0 < q < 1/2$, then $2q^3 - q^2 - (2q^3 - 3q^2 + q)S = q(2q-1)[q - (q-1)S] < 0$. Therefore, $\Delta > 0$ always holds. It also holds that $T_3 < \frac{1-(S+1)q}{1-q}$. Therefore,

$$\left\{ T : \Delta > 0, T < \frac{1-(S+1)q}{1-q}, T < T_3 \right\} = \{ T : T < T_3 \}.$$

Thus, when $(q-1)S > q$, the probability p_3 that the replicator-mutator dynamics has three equilibria is given by

$$p_3 = \mathbb{P}(T < T_3) = 1 - \mathbb{P}(T > T_3) = 1 - p_2 = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{T_3 - T_0}{\sigma\sqrt{2}}\right).$$

It implies that in this case, $p_1 = 0$. Note that, SD and H games always satisfy this case since $s_2 = \frac{q}{q-1} < 0 < S$. Thus, we obtain for SD and H games:

$$p_1 = 0, \quad p_2 = \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{(q-1)S/q - T_0}{\sigma\sqrt{2}}\right), \\ p_3 = 1 - p_2 = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{(q-1)S/q - T_0}{\sigma\sqrt{2}}\right).$$

(C2) $S = s_2$

Then $2q^3 - q^2 - (2q^3 - 3q^2 + q)S = 0$, we obtain that

$$T \neq -\frac{q(1-q)S + q^2 + 2q - 1}{(1-q)^2} \Leftrightarrow \Delta > 0. \tag{16}$$

A direct computation shows that $-\frac{q(1-q)S + q^2 + 2q - 1}{(1-q)^2} < T_3$. Thus, the probability that the replicator-mutator equation has 3 equilibria is again $p_3 = \mathbb{P}(T < T_3)$. Thus we obtain the same results as in the previous case.

(C3) $s_1 < S < s_2$

In this case since $2q^3 - q^2 - (2q^3 - 3q^2 + q)S > 0$, $\Delta = 0$ has two distinct solutions T_1 and T_2 . In addition, we have

$$\frac{1-(S+1)q}{1-q} > T_3 > T_2.$$

It implies that the probability of the replicator-mutator equation having three equilibria is

$$p_3 = \mathbb{P}(T < T_1) + \mathbb{P}(T_2 < T < T_3) \\ = \int_{-\infty}^{T_1} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-T_0)^2}{2\sigma^2}\right) dx + \int_{T_2}^{T_3} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-T_0)^2}{2\sigma^2}\right) dx \\ = \frac{1}{2} \left[1 + \operatorname{erf}\left(\frac{T_1 - T_0}{\sigma\sqrt{2}}\right) + \operatorname{erf}\left(\frac{T_3 - T_0}{\sigma\sqrt{2}}\right) - \operatorname{erf}\left(\frac{T_2 - T_0}{\sigma\sqrt{2}}\right) \right].$$

(C4) $S \leq s_1$

Similarly to case (C3), $\Delta = 0$ has two distinct solutions T_1 and T_2 and

$$\frac{1-(S+1)q}{1-q} \leq T_3, \quad T_1 \leq \frac{1-(S+1)q}{1-q} \leq T_2.$$

Thus the probability that replicator-mutator equation has three equilibria is given by

$$p_3 = \mathbb{P}(T < T_1) \\ = \int_{-\infty}^{T_1} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-T_0)^2}{2\sigma^2}\right) dx$$

$$= \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{T_1 - T_0}{\sigma\sqrt{2}}\right).$$

Bringing all cases together, we obtain, in PD and SH games, the probability that the replicator-mutator equation has three equilibria is

$$p_3 = \begin{cases} \frac{1}{2} \left[1 + \operatorname{erf}\left(\frac{T_1 - T_0}{\sigma\sqrt{2}}\right) \right], & \text{if } S \leq s_1, \\ \frac{1}{2} \left[1 + \operatorname{erf}\left(\frac{T_1 - T_0}{\sigma\sqrt{2}}\right) + \operatorname{erf}\left(\frac{T_3 - T_0}{\sigma\sqrt{2}}\right) \right], & \text{if } s_1 < S < s_2, \\ -\operatorname{erf}\left(\frac{T_2 - T_0}{\sigma\sqrt{2}}\right), & \text{if } S \geq s_2. \end{cases}$$

The three cases in the formula above can be written in terms of q as follows

$$p_3 = \begin{cases} \frac{1}{2} \left[1 + \operatorname{erf}\left(\frac{T_1 - T_0}{\sigma\sqrt{2}}\right) \right], & \text{if } 0 \leq q \leq q_1, \\ \frac{1}{2} \left[1 + \operatorname{erf}\left(\frac{T_1 - T_0}{\sigma\sqrt{2}}\right) + \operatorname{erf}\left(\frac{T_3 - T_0}{\sigma\sqrt{2}}\right) \right], & \text{if } q_1 < q < q_2, \\ -\operatorname{erf}\left(\frac{T_2 - T_0}{\sigma\sqrt{2}}\right), & \text{if } q_2 \leq q \leq \frac{1}{2}, \end{cases}$$

where q_1 and q_2 are respectively unique solutions of

$$\frac{q(q-1)}{1-2q} = S \quad \text{and} \quad \frac{q}{q-1} = S.$$

We note that when $q_2 \leq q \leq \frac{1}{2}$ then $p_3 = 1 - p_2$, thus $p_1 = 0$. Using the above formula, in principle, we can derive qualitative properties for p_3 as a function of q as in Lemma 1. However, to compute the derivatives of T_1 and T_2 with respect to q and determine their signs for general S are very complicated for general S . In the following example, we demonstrate such a result for the specific value $S = -0.5$.

Example 1. For $S = -0.5$, we obtain the following formula for p_3 depending on the value of q

$$p_3 = \begin{cases} \frac{1}{2} \left[1 + \operatorname{erf}\left(\frac{T_1 - T_0}{\sigma\sqrt{2}}\right) \right], & \text{if } q \leq \frac{2-\sqrt{2}}{2}, \\ \frac{1}{2} \left[1 + \operatorname{erf}\left(\frac{T_1 - T_0}{\sigma\sqrt{2}}\right) + \operatorname{erf}\left(\frac{T_3 - T_0}{\sigma\sqrt{2}}\right) \right], & \text{if } \frac{2-\sqrt{2}}{2} \leq q \leq \frac{1}{3}, \\ -\operatorname{erf}\left(\frac{T_2 - T_0}{\sigma\sqrt{2}}\right), & \text{if } \frac{1}{3} \leq q \leq \frac{1}{2}. \end{cases}$$

From the above analytical formula, we can study the behaviour of p_3 as a function of q . For $0 < q < \frac{2-\sqrt{2}}{2}$, we have

$$\frac{dp_3}{dq} = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(T_1 - T_0)^2}{2\sigma^2}\right] \frac{dT_1}{dq} < 0.$$

For $\frac{2-\sqrt{2}}{2} < q < \frac{1}{3}$,

$$\begin{aligned} \frac{dp_3}{dq} &= \frac{1}{\sigma\sqrt{2\pi}} \left(\exp\left[-\frac{(T_1 - T_0)^2}{2\sigma^2}\right] \frac{dT_1}{dq} + \exp\left[-\frac{(T_3 - T_0)^2}{2\sigma^2}\right] \frac{dT_3}{dq} \right. \\ &\quad \left. - \exp\left[-\frac{(T_2 - T_0)^2}{2\sigma^2}\right] \frac{dT_2}{dq} \right) > 0. \end{aligned}$$

For $\frac{1}{3} < q < \frac{1}{2}$,

$$\frac{dp_3}{dq} = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(T_3 - T_0)^2}{2\sigma^2}\right] \frac{dT_3}{dq} < 0$$

Thus as a function of q , p_3 is decreasing in $(0, \frac{2-\sqrt{2}}{2})$, is increasing in $(\frac{2-\sqrt{2}}{2}, \frac{1}{3})$ and then is decreasing again in $(\frac{1}{3}, \frac{1}{2})$.

3.4. Numerical simulations

In Fig. 1 we plot the probabilities that PD, SD, SH and H games have 1, 2 or 3 equilibria for different values of q using the analytical formula obtained in the previous section (see Theorem 1). For validation, these probabilities are also computed by sampling over 10^6 realisations of T from the normal distribution $\mathcal{N}(T_0, 1)$ and then calculating the solutions of $g(t) = 0$. In these simulations, the value S_0 (respectively, T_0) is taken to be the middle point in the interval that S (respectively, T) belongs to in each corresponding game, that is $S_0 = -0.5$ in PD and SH games and $S_0 = 0.5$ in SD and H games (respectively, $T_0 = 1.5$ in PD and SD games, $T_0 = 0.5$ in SH and H games). It can be clearly seen that the simulation results are in accordance with the analytical ones. In particular, we observe that, as a function of q , the probability p_2 is decreasing in SD and H games, and is increasing in PD and SH games.

In Fig. 2 we plot the probabilities that PD, SD, SH and H games have 1, 2 or 3 equilibria for different values of σ (fixing $q = 0.25$), using the analytical formula obtained in the previous section. The value S_0 (respectively, T_0) is taken to be the middle point in the interval that S (respectively, T) belongs to in each corresponding game, that is $S_0 = -0.5$ in PD and SH games and $S_0 = 0.5$ in SD and H games (respectively, $T_0 = 1.5$ in PD and SD games, $T_0 = 0.5$ in SH and H games).

4. S is random

In this section, we consider the case where only S is random, assuming that

$$S = S_0 + \varepsilon_S,$$

where $\varepsilon_S \sim N(0, \eta^2)$. Then we have

$$S \sim N(S_0, \eta^2).$$

As in Section 3, we will compute the probability that the replicator-mutator dynamics has $k \in \{1, 2, 3\}$ equilibria, which is the same as the probability that the polynomial g defined in (9) has $k - 1 \in \{0, 1, 2\}$ positive roots. This has been characterised in three corresponding cases (a), (b) and (c) in Section 3.1.

4.1. The probability that the replicator-mutator has two equilibria

In this section, we compute the probability that the replicator-mutator dynamics has 2 equilibria, which amounts to computing the probability that the quadratic polynomial g defined in (9) has 1 positive root. This can happen in two different cases below.

(C1) $\Delta = 0$, $q + a - c < 0$.

Let $\hat{T} := T - 1$ and

$$X(\hat{T}, S) := q + a - c = (1 - q)(T_0 - 1) + qS, \tag{17}$$

$$Y(\hat{T}, S) := c - q = q(T_0 - 1) + (1 - q)S. \tag{18}$$

Then, the equation $\Delta = 0$ can be rewritten as

$$X^2 + 4qY = -4q^2. \tag{19}$$

Substituting X and Y in (17) and (18) to (19), we get

$$S = \frac{-(1 - q)(T_0 + 1) \pm 2\sqrt{T_0(1 - 2q)}}{q},$$

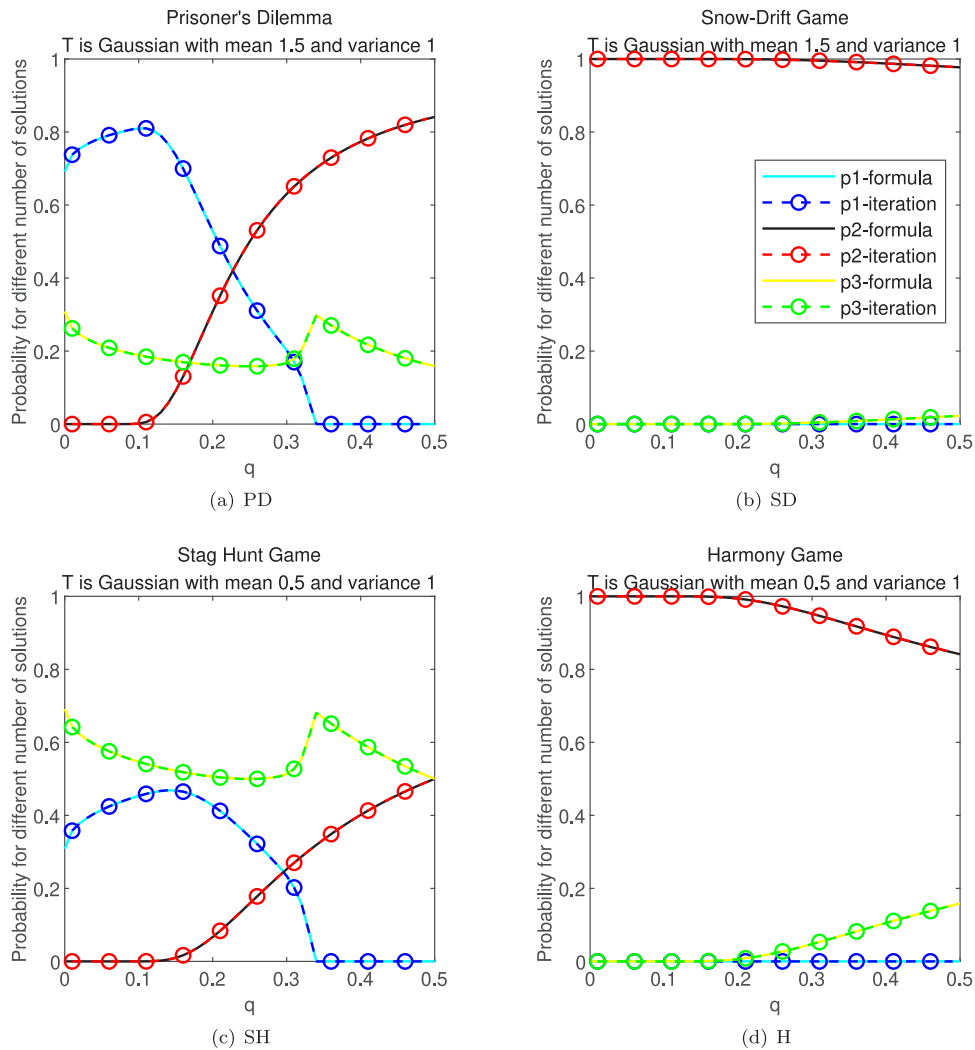


Fig. 1. Probabilities p_1, p_2, p_3 that the replicator-mutator dynamics has respectively 1, 2, 3 equilibria as functions of the mutation rate q in PD, SD, SH and H games when T is random and S is fixed. The values from analytical analysis and numerical samplings are in accordance.

with $\frac{-1-(T_0-1)q}{1-q} < S < \frac{-qT_0}{1-q}$. Since S is a continuous random variable, the probability that this case occurs is 0.

(C2) $\Delta > 0$ and $c \geq 0$.

It follows from (18) that $c \geq 0$ is equivalent to $Y \geq -q$, which gives

$$S \geq \frac{-qT_0}{1-q}.$$

From (17) and (18), we have

$$\begin{aligned} \Delta &= X^2 + 4qY + 4q^2 \\ &= ((1-q)(T_0-1) + qS)^2 + 4q(q(T_0-1) + (1-q)S) + 4q^2 \\ &= q^2S^2 + 2q(1-q)(T_0+1)S + 4q^2T_0 + (1-q)^2(T_0-1)^2. \end{aligned}$$

Thus Δ can be seen as a quadratic polynomial of S with a leading coefficient $q^2 > 0$ and its discriminant is given by

$$\hat{\Delta} = 16q^2T_0(1-2q) > 0$$

since for $0 < q < \frac{1}{2}$. Thus The equation $\Delta = 0$ have two real solutions, $S_1 < S_2$, given by

$$S_1 = \frac{(q-1)(T_0+1) - 2\sqrt{T_0-2qT_0}}{q}, \quad S_2 = \frac{(q-1)(T_0+1) + 2\sqrt{T_0-2qT_0}}{q}. \quad (20)$$

To proceed, we need to compare $\frac{-qT_0}{1-q}$ with S_2 . We have

$$\begin{aligned} \frac{\frac{-qT_0}{1-q}}{S_2} &= \frac{-qT_0}{1-q} \frac{q}{(q-1)(T_0+1) + 2\sqrt{T_0-2qT_0}} \\ &= \frac{q^2T_0}{q^2T_0 + (q-1)^2 + (1-2q)T_0 + 2\sqrt{T_0-2qT_0}} < 1. \end{aligned}$$

Since both $\frac{-qT_0}{1-q}$ and S_2 are negative, it follows that $\frac{-qT_0}{1-q} > S_2$. Therefore

$$\mathbb{P}(\Delta > 0, Y \geq -q) = \mathbb{P}(S > \frac{-qT_0}{1-q}).$$

Since $S \sim \mathcal{N}(S_0, \eta^2)$, the probability that g has a unique positive root, which is the probability that the replicator-mutator dynamics have two equilibria, is given by

$$\begin{aligned} p_2 &= \int_{\frac{-qT_0}{1-q}}^{\infty} \frac{1}{\eta\sqrt{2\pi}} e^{-\frac{(x-S_0)^2}{2\eta^2}} dx \\ &= \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{\frac{-qT_0}{1-q} - S_0}{\sqrt{2}\eta}\right). \end{aligned} \quad (21)$$

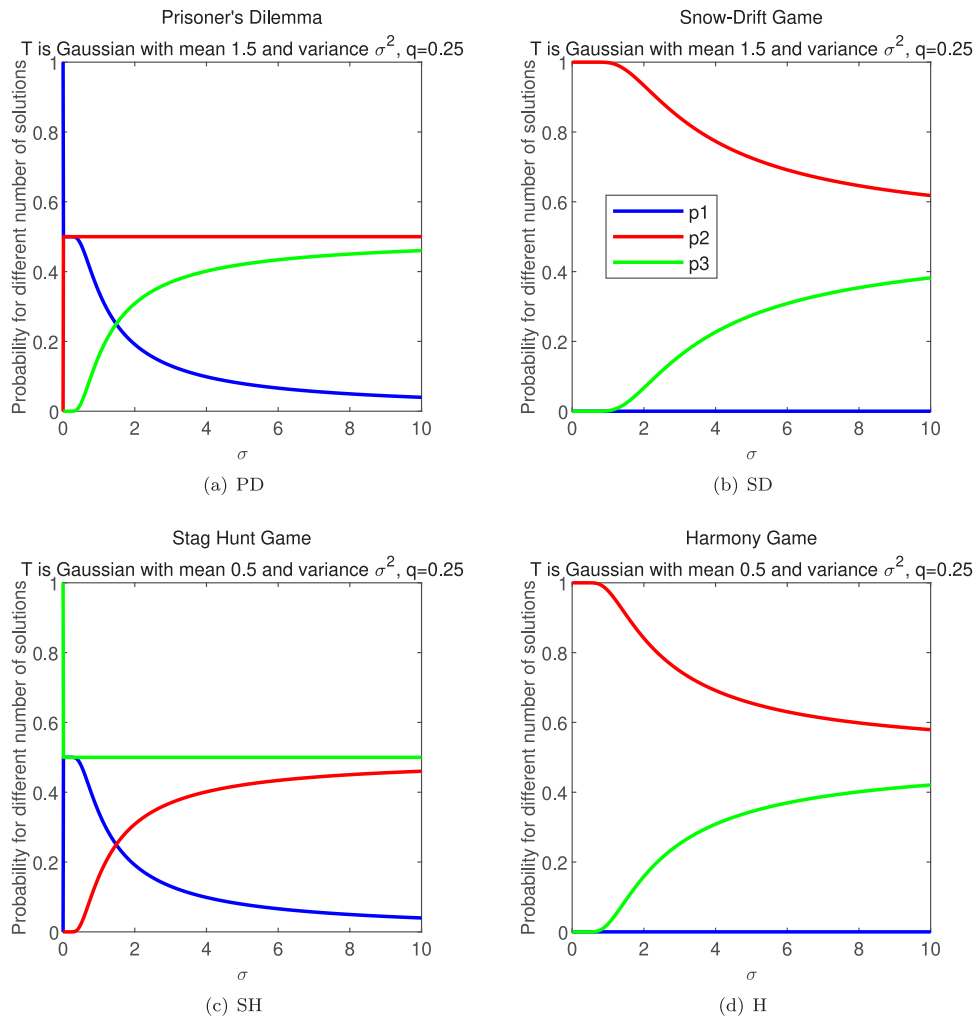


Fig. 2. The probabilities p_1 , p_2 and p_3 that the replicator-mutator dynamics has respectively 1, 2 and 3 equilibria as functions of σ , the variance of the random variable T , in PD, SD, SH and H games when T is random and S is fixed.

Lemma 2. As a function of q , the probability p_2 is increasing in all games. As a consequence,

$$\frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{-S_0}{\sqrt{2}\eta}\right) < p_2 < \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{-T_0 - S_0}{\sqrt{2}\eta}\right).$$

Proof. Let $z := \frac{-qT_0 - S_0}{\eta\sqrt{2}}$. Then $p_2 = \frac{1}{2} - \frac{1}{2} \operatorname{erf}(z)$. Thus, since $T_0 > 0$, we have

$$\frac{dp_2}{dq} = \frac{dp_2}{dz} \frac{dz}{dq} = \frac{1}{\sqrt{\pi}} e^{-z^2} \frac{T_0}{\eta\sqrt{2}(1-q)^2} > 0.$$

Thus p_2 is increasing as a function of q , see Fig. 3. As a consequence, we obtain the following lower and upper estimates for p_2 :

$$p_2|_{q=0} = \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{-S_0}{\sqrt{2}\eta}\right) < p_2 < p_2|_{q=1/2} = \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{-T_0 - S_0}{\sqrt{2}\eta}\right). \quad \square$$

The following example illustrates the above lemma.

Example 2. In the SD game, with the specific values $S = 0.5, T_0 = 1.5, \eta = 1$ we get

$$\frac{1}{2} - \frac{1}{2} \operatorname{erf}(-0.5/\sqrt{2}) = 0.691462 < p_2 < \frac{1}{2} - \frac{1}{2} \operatorname{erf}(-2/\sqrt{2}) = 0.97725.$$

In the PD game, with $T_0 = 1.5, S_0 = -0.5, \eta = 1$, we get

$$\frac{1}{2} - \frac{1}{2} \operatorname{erf}(0.5/\sqrt{2}) = 0.308538 < p_2 < \frac{1}{2} - \frac{1}{2} \operatorname{erf}(-1/\sqrt{2}) = 0.8413.$$

In the SH game, with $T_0 = 0.5, S = -0.5, \eta = 1$, we get

$$\frac{1}{2} - \frac{1}{2} \operatorname{erf}(0.5/\sqrt{2}) = 0.308538 < p_2 < \frac{1}{2} - \frac{1}{2} \operatorname{erf}(0) = 0.5.$$

In the H game, with $T_0 = S_0 = 0.5, \eta = 1$, we get

$$\frac{1}{2} - \frac{1}{2} \operatorname{erf}(-0.5/\sqrt{2}) = 0.691462 < p_2 < \frac{1}{2} - \frac{1}{2} \operatorname{erf}(-1/\sqrt{2}) = 0.8413.$$

We notice that in SD and H games, p_2 is always dominant.

4.2. The probability that the replicator-mutator has three equilibria

In this section, we compute the probability p_3 that the replicator-mutator has three equilibria, that is the probability that g has two positive roots. We have

$$\begin{aligned} p_3 &= \mathbb{P}(\Delta > 0, c < 0, (-q - a + c) > 0) = \mathbb{P}(\Delta > 0, Y < -q, X < 0) \\ &= \mathbb{P}(\Delta > 0, S < \frac{(q-1)(T_0-1)}{q}, S < \frac{-qT_0}{1-q}). \end{aligned}$$

To proceed, by comparing $\frac{(q-1)(T_0-1)}{q}$ with $\frac{-qT_0}{1-q}$ and with S_1, S_2 , we obtain 2 cases.

(i) When $T_0 > \frac{(1-q)^2}{1-2q}$, then we have

$$\frac{-qT_0}{1-q} > S_2 > \frac{(q-1)(T_0-1)}{q} > S_1.$$

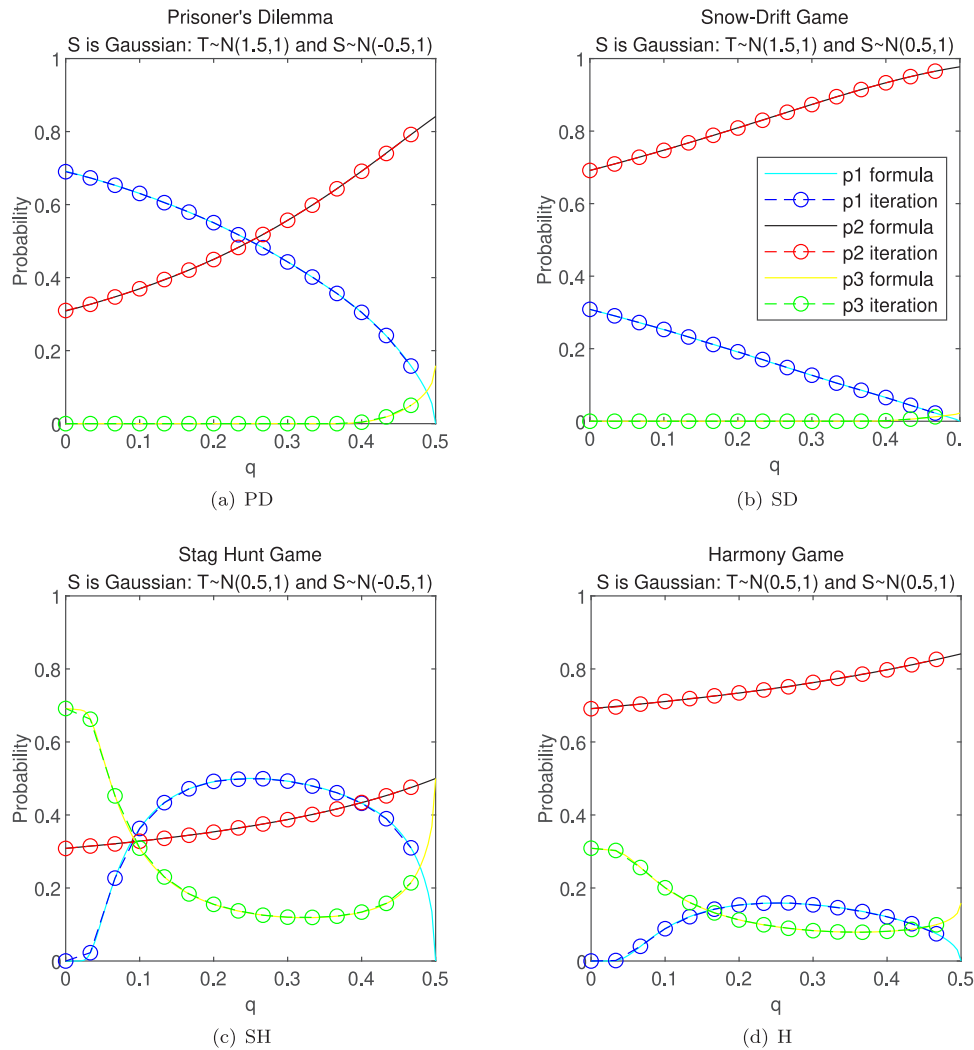


Fig. 3. Probabilities p_1, p_2, p_3 that the replicator-mutator dynamics has respectively 1, 2, 3 equilibria, as functions of the mutation strength q in PD, SD, SH and H games when S is random and T is fixed. The values from analytical analysis and numerical samplings are in accordance.

Thus, the probability p_3 in this case is given by

$$p_3 = \mathbb{P}(S < S_1).$$

(ii) When $T_0 < \frac{(1-q)^2}{1-2q}$, then we have

$$\frac{(q-1)(T_0-1)}{q} > \frac{-qT_0}{1-q} > S_2 > S_1.$$

Thus, the probability p_3 in this case is given by

$$p_3 = \mathbb{P}(S < S_1) + \mathbb{P}(S_2 < S < \frac{-qT_0}{1-q}).$$

Therefore, since $S \sim \mathcal{N}(S_0, \eta^2)$, the probability of having three equilibria is given by

$$p_3 = \begin{cases} \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{S_1 - S_0}{\sqrt{2}\eta}\right), & \text{if } T_0 > \frac{(1-q)^2}{1-2q}, \\ \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{S_1 - S_0}{\sqrt{2}\eta}\right) + \frac{1}{2} \operatorname{erf}\left(\frac{\frac{-qT_0}{1-q} - S_0}{\sqrt{2}\eta}\right) - \frac{1}{2} \operatorname{erf}\left(\frac{S_2 - S_0}{\sqrt{2}\eta}\right), & \text{if } T_0 < \frac{(1-q)^2}{1-2q}. \end{cases}$$

In conclusion:

$$p_2 = \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{\frac{qT_0}{1-q} - S_0}{\sqrt{2}\eta}\right),$$

$$p_3 = \begin{cases} \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{S_1 - S_0}{\sqrt{2}\eta}\right), & \text{if } T_0 > \frac{(1-q)^2}{1-2q}, \\ \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{S_1 - S_0}{\sqrt{2}\eta}\right) + \frac{1}{2} \operatorname{erf}\left(\frac{\frac{-qT_0}{1-q} - S_0}{\sqrt{2}\eta}\right) - \frac{1}{2} \operatorname{erf}\left(\frac{S_2 - S_0}{\sqrt{2}\eta}\right), & \text{if } T_0 < \frac{(1-q)^2}{1-2q}. \end{cases}$$

$$p_1 = 1 - p_2 - p_3.$$

4.3. Numerical simulations

In Fig. 3 we show the probabilities that PD, SD, SH and H games have 1, 2 or 3 equilibria for different values of q using the analytical formula obtained in the previous section (see Theorem 2). Moreover, for validation, these probabilities were calculated by sampling over 10^6 realisations of S from the normal distribution $\mathcal{N}(S_0, 1)$ and then calculating the solutions of $g(t) = 0$. The values of T_0 and S_0 are

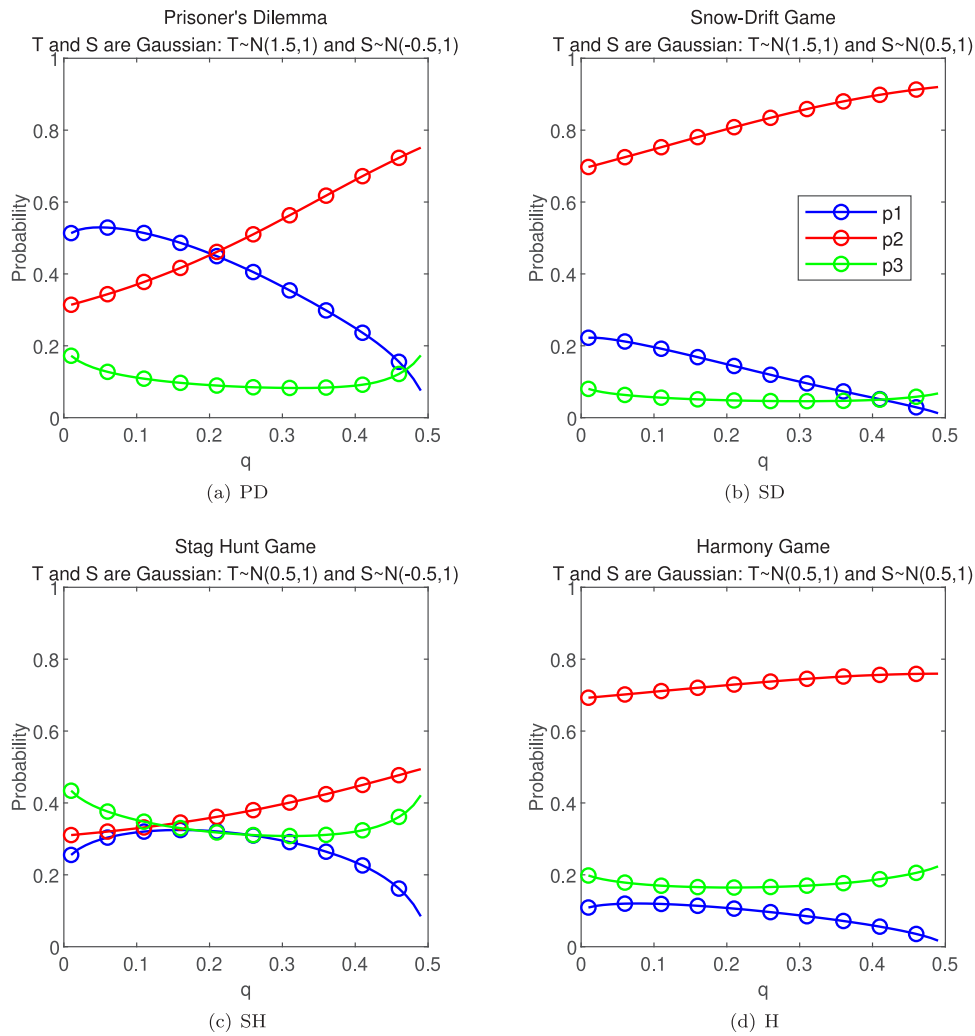


Fig. 4. Probabilities p_1, p_2, p_3 that the replicator-mutator dynamics has respectively 1, 2, 3 equilibria, as functions of the mutation strength q in PD, SD, SH and H games when both T and S are random. The values are numerically obtained from 10^6 samplings.

the middle points of the corresponding intervals as in Section 3.4. It can be seen that the numerical results are clearly in accordance with theoretical ones. We can also observe that, p_2 is always increasing as a function of q , as stated in Theorem 2.

5. Both T and S are random: numerical investigations

In this section, we numerically compute the probabilities p_1, p_2 and p_3 when both T and S are random. In Fig. 4, we show their values obtained from averaging over 10^6 random samples of T and S with the corresponding distribution in each game. Namely, T and S have Gaussian distributions with means chosen corresponding to one of the social dilemma games. In our simulations, for PD, $T \sim N(1.5, 1)$ and $S \sim N(-0.5, 1)$; for SD, $T \sim N(1.5, 1)$ and $S \sim N(0.5, 1)$; for SH, $T \sim N(0.5, 1)$ and $S \sim N(-0.5, 1)$; and, for H game, $T \sim N(0.5, 1)$ and $S \sim N(0.5, 1)$. We observe that p_2 tends to increase in all games, while p_1 and p_3 exhibit more complex behaviours. We aim to study this more complex case analytically in future work.

6. Summary and outlook

In this paper, we have studied pair-wise social dilemmas where the payoff entries are random variables. The randomness is necessary to capture the uncertainty that is unavoidable in practical applications, which may come from different sources, both subjective and

objective, such as lack of data, fluctuating environment as well as human estimate errors. We have focused on four important social dilemma games, namely Prisoner’s Dilemma, the Snow-Drift game, the Stag-Hunt game and the Harmony game. For each game, we have analytically computed, and numerically validated, the probability that the replicator-mutator dynamics has a certain number of equilibria, studying their qualitative behaviour as a function of the mutation rate. Our results have clearly shown that the mutation rate and randomness from the payoff matrix have a strong impact on the equilibrium outcomes. Thus, our analysis has provided novel theoretical contributions to the understanding of the impact of uncertainty on the behavioural diversity in a complex dynamical system.

Here we have assumed that the payoff entries are standard normal distributions; however, from formulas such as Eq. (14), our results can be easily extended to other distributions. One natural and challenging problem for future work is to generalise the equilibrium analysis of the present work to multi-player and multi-strategy games where the payoff entries satisfy more complex conditions. Another direction is to trajectoryally characterise statistical properties of the full dynamical systems.

CRediT authorship contribution statement

Luor Chen: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Formal analysis, Data curation.

Churou Deng: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Investigation, Formal analysis, Data curation. **Manh Hong Duong:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization. **The Anh Han:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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